FUNCTORIAL PROPERTIES OF THE HYPERGEOMETRIC MAP

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Abstract. The quantized Knizhnik-Zamolodchikov equation is a difference equation defined in terms of rational R matrices. We describe all singularities of hypergeometric solutions to the qKZ equations.

1. Introduction

The quantized Knizhnik-Zamolodchikov equation (qKZ) is a system of difference equations. The qKZ equation was introduced in [FR] as an equation for matrix elements of vertex operators of a quantum affine algebra. A special case of the qKZ equation had been introduced earlier in [S] as equations for form factors in integrable quantum field theory. Later, the qKZ equation was derived as an equation for correlation functions in lattice integrable models, cf. [JM] and references therein.

In this paper we consider the rational qKZ equation associated with $sl(2)$. The qKZ equation with values in a tensor product of $sl(2)$ Verma modules $V(\lambda_1)\otimes\ldots\otimes V(\lambda_n)$ was solved in [TV], cf. [M], [R]. The solutions $\Psi(z,\lambda)$, $z=(z_1,\ldots,z_n)$, $\lambda=(\lambda_1,\ldots,\lambda_n)$ are meromorphic functions written in terms of hypergeometric integrals, see (16). Here $V(\lambda_i)$ is the Verma module with highest weight $\lambda_i\in\mathbb{C}$. We endow the module $V(\lambda_i)$ with an evaluation Yangian module structure with complex evaluation parameter z_i and denote $V(z_i,\lambda_i)$. Set $V(z,\lambda)=V(z_1,\lambda_1)\otimes\ldots\otimes V(z_n,\lambda_n)$. The space of the hypergeometric solutions can be naturally identified with the corresponding tensor product of Verma modules $V_q(\lambda_1)\otimes\ldots\otimes V_q(\lambda_n)$ over the quantum group $U_q sl(2)$, where $q=\exp(\pi i/p)$ and p is the step of the qKZ equation. We endow the module $V_q(\lambda_i)$ with an evaluation structure of affine quantum group $U_q\widehat{gl(2)}$ module with complex evaluation parameter z_i and denote $V_q(z_i,\lambda_i)$. Set $V_q(z,\lambda)=V_q(z_1,\lambda_1)\otimes\ldots\otimes V_q(z_n,\lambda_n)$.

Thus, the hypergeometric solutions define a (hypergeometric) map

$$\mathrm{qKZ}(z,\lambda;p):\ V_q(z,\lambda)\to V(z,\lambda),$$

see (17).

We call parameters p,z,λ generic if the $U_q\widehat{gl(2)}$ module $V_q(z,\lambda)$ is irreducible. For generic values of parameters, the hypergeometric map is an isomorphism of vector spaces.

The Yangian module $V(z,\lambda)$ is reducible iff $z_a-z_b+\lambda_a+\lambda_b\in\mathbb{Z}_{\geqslant 0}$ for some $a,b\in\{1,\ldots,n\}$. We describe the properties of the hypergeometric map for these values of parameters.

Case 1. Let $2\lambda_1=k\in\mathbb{Z}_{\geqslant 0}$. The module $V(z,\lambda)$ has a submodule isomorphic to $V(z,\lambda')$, where $\lambda'=(-\lambda_1-1,\lambda_2,\ldots,\lambda_n)$. The module $V_q(z,\lambda)$ has a submodule isomorphic to $V_q(z,\lambda')$. Then the qKZ map is still a well defined isomorphism of vector spaces.

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Moreover, it maps the $U_q\widehat{gl(2)}$ submodule $V_q(z,\lambda')$ onto the Yangian submodule $V(z,\lambda')$ and the restriction of the map $\mathrm{qKZ}(z,\lambda)$ to $V_q(z,\lambda')$ coincides up to a scalar factor with the hypergeometric map $\mathrm{qKZ}(z,\lambda')$, see Theorem 29.

Case 2. Let $z_2 - z_1 + \lambda_1 + \lambda_2 = k \in \mathbb{Z}_{\geqslant 0}$. The module $V_q(z,\lambda)$ has a submodule isomorphic to $V_q(z',\lambda')$, where $z',\lambda' \in \mathbb{C}^n$ are some new values of parameters. The module $V(z,\lambda)$ has a nontrivial submodule such that the factor module is isomorphic to $V(z',\lambda')$. Then, the qKZ map is a well defined linear map. The kernel of the qKZ map coincides with the submodule $V_q(z',\lambda')$ and the image of the qKZ map coincides with the proper submodule in $V(z,\lambda)$, see Corollary 31.

Case 3. Let $z_1 - z_2 + \lambda_1 + \lambda_2 = k \in \mathbb{Z}_{\geqslant 0}$. The module $V(z,\lambda)$ has a submodule isomorphic to $V(z',\lambda')$, where $z',\lambda' \in \mathbb{C}^n$ are some new values of parameters. The module $V_q(z,\lambda)$ has a nontrivial submodule such that the factor module is isomorphic to $V_q(z',\lambda')$. Then the qKZ map has a simple pole at the hyperplane $z_1 - z_2 + \lambda_1 + \lambda_2 = k$. Let Res be the corresponding residue of the qKZ map. The kernel of Res coincides with the proper submodule of $V_q(z,\lambda)$ and the image of Res coincides with the proper submodule of $V(z,\lambda)$. Thus, Res is a linear isomorphism of the factor module $V_q(z',\lambda')$ to the submodule $V(z',\lambda')$. We prove that the map Res coincides with the hypergeometric map $\mathrm{qKZ}(z',\lambda')$ up to a scalar factor, see Example 33.

It is well known that the intertwinings of Yangian modules and of modules over $U_q\widehat{gl(2)}$ possess similar properties, see Lemmas 9-11 and Lemmas 21-23.

Let $V(\tilde z,\tilde\lambda)$ be a Yangian module obtained from the tensor product $V(z,\lambda)$ by a permutation of factors. It is well known that $V(\tilde z,\tilde\lambda) \simeq V(z,\lambda)$, $V_q(\tilde z,\tilde\lambda) \simeq V_q(z,\lambda)$ and the qKZ map intertwines these isomorphisms .

Let $z,\lambda \in \mathbb{C}^n$. The values of parameters $z',\lambda' \in \mathbb{C}^n$ such that $V(z,\lambda) \simeq V(z',\lambda')$ are parametrized by pairs of permutations $\sigma,\sigma' \in \mathbb{S}^n$. We show that the qKZ map intertwines all these isomorphisms as well, see the diagram in Theorem 25. This observation allows us to reduce cases 2 and 3 to case 1.

We apply the above observation to include the qKZ equation in a bigger compatible system of difference equations which we call the extended qKZ equation. The shifts of the extended qKZ equation generate a group acting in the space $\mathbb{C}^{2n}$ of parameters (z,λ) isomorphic to $\mathbb{Z}^{2n-1}$.

We use the above study to describe all singularities of the hypergeometric solutions, see Theorem 34, and the Remark after it.

In this paper we treat the case $|\kappa| > 1$, where κ is a parameter of the qKZ equation (15). Our results can be carried in the same fashion in the case $\kappa = 1$. We also obtain similar results in the case of tensor products of finite dimensional representations, see Remarks after Theorem 34.

The paper is organized as follows. We study the functional model of the Yangian module $V(z,\lambda)$ in Section 2. In Sections 2.1-2.4 we fix our notations and recall some technical facts from [TV]. In Sections 2.5-2.7 we describe the maps of Yangian modules in terms of spaces of functions. We study the functional model of the $U_q\widehat{gl(2)}$ module $V_q(z,\lambda)$ in Section 3. Section 3 is constructed similarly to Section 2. In Sections 4.1-4.2 we define the qKZ equation and the qKZ map. In Section 4.3 we introduce the extended qKZ equation. Section 4.5 contains our main results, Theorem 29 and Corollaries 30-32. We describe the singularities of the hypergeometric map in Section 4.6.

2. Rational hypergeometric space of functions

2.1. **The Lie algebra $sl(2)$.** Let e, f, h be generators of the Lie algebra $sl(2)$ such that

$$[h,e]=e, \qquad [h,f]=-f, \qquad [e,f]=2h.$$

For an $sl(2)$ module M, let M^* be its restricted dual with an $sl(2)$ module structure defined by

$$\langle e\varphi, x\rangle = \langle \varphi, fx\rangle, \qquad \langle f\varphi, x\rangle = \langle \varphi, ex\rangle, \qquad \langle h\varphi, x\rangle = \langle \varphi, hx\rangle,$$

for all $x \in M$, $\varphi \in M^*$.

For a complex number λ, denote $V(\lambda)$ the $sl(2)$ Verma module with highest weight λ and highest vector v. The vectors $\{f^l v,\ l \in \mathbb{Z}_{\geqslant 0}\}$ form a basis in $V(\lambda)$. The Shapovalov form on $V(\lambda)$ is a bilinear form B_λ such that

$$B_\lambda(f^l v, f^l v) = l! \prod_{a=0}^{l-1}(2\lambda - a), \qquad B_\lambda(f^l v, f^k v) = 0,\ l \neq k. \tag{1}$$

The Shapovalov form B on a tensor product $V(\lambda_1)\otimes\ldots\otimes V(\lambda_n)$ is defined by $B = B_{\lambda_1}\otimes\ldots\otimes B_{\lambda_n}$. The dual map to the Shapovalov form defines a map of $sl(2)$ modules

$$Sh:\ V(\lambda_1)\otimes\ldots\otimes V(\lambda_n) \to \left(V(\lambda_1)\otimes\ldots\otimes V(\lambda_n)\right)^*.$$

For $\lambda \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$, denote $L(\lambda)$ the $(2\lambda+1)$- dimensional irreducible $sl(2)$ module.

2.2. **The Yangian $Y(gl(2))$.** The Yangian $Y(gl(2))$ is an associative algebra with an infinite set of generators $T_{i,j}^{(a)}$, $i,j=1,2$, $a=0,1,\ldots$, subject to the relations:

$$R(x-y)T_{(1)}(x)T_{(2)}(y) = T_{(2)}(y)T_{(1)}(x)R(x-y),$$

where $R(x) = (x\,\mathrm{Id} + P) \in \mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2)$, $P \in \mathrm{End}(\mathbb{C}^2\otimes\mathbb{C}^2)$ is the operator of permutation of the two factors and $T_{ij}(u) = \sum\limits_{s=0}^{\infty} T_{ij}^{(a)} u^{-a}$ are generating series, $T_{(1)}(x) = 1\otimes T(x)$, $T_{(2)}(x) = T(x)\otimes 1$.

In this paper we take tensor products and dual of Yangian modules using the comultiplication $\Delta : Y(gl(2)) \to Y(gl(2))\otimes Y(gl(2))$ and the antipode $S : Y(gl(2)) \to Y(gl(2))$ given by

$$\Delta\ :\ T_{ij}(u) \mapsto \sum_{a=1}^{2} T_{ia}(u)\otimes T_{aj}(u), \tag{2}$$

$$S\ :\ T_{ij}(u) \mapsto T_{ji}(u), \qquad i,j=1,2.$$

For a complex number z, there is an automorphism $\rho_z : Y(gl(2)) \to Y(gl(2))$ of the form

$$\rho_z\ :\ T_{ij}(u) \mapsto T_{ij}(u-z), \qquad i,j=1,2.$$

The evaluation morphism $\epsilon : Y(gl(2)) \to U(sl(2))$, to the universal enveloping algebra of $sl(2)$, has the form

$$\epsilon\ :\ T_{11}(u) \mapsto h/u, \qquad \epsilon\ :\ T_{12}(u) \mapsto f/u,$$

$$\epsilon\ :\ T_{21}(u) \mapsto e/u, \qquad \epsilon\ :\ T_{22}(u) \mapsto -h/u.$$

For complex numbers z,λ, denote $V(z,\lambda)$ the $sl(2)$ Verma module $V(\lambda)$ endowed with a structure of Yangian module via pull back with respect to the map $\epsilon\circ\rho_z$. The module $V(z,\lambda)$ is called *the evaluation Verma module.*

For $\lambda \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$, denote $L(z,\lambda)$ the $sl(2)$ module $L(\lambda)$ endowed with a structure of Yangian module via pull back with respect to the map $\epsilon \circ \rho_z$. The module $L(z,\lambda)$ is called *the evaluation finite dimensional module.*

For more detail on the Yangian see [CP], [TV].

2.3. **The rational hypergeometric space.** Fix a natural number n and $z = (z_1, \dots, z_n)$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$.

Define *a rational hypergeometric space* $\mathfrak{F}(z,\lambda) = \bigoplus_{l=0}^{\infty} \mathfrak{F}_l(z,\lambda)$, where $\mathfrak{F}_l(z,\lambda)$ is the space of functions of the form

$$P(t_1,\dots,t_l) \prod_{a=1}^{n}\prod_{b=1}^{l} \frac{1}{t_b - z_a - \lambda_a} \prod_{1\leqslant a<b\leqslant l} \frac{t_a - t_b}{t_a - t_b + 1}\,.$$

Here P is a polynomial with complex coefficients which is symmetric in variables $t_1, \dots, t_l$ and has degree less than n in each of variables $t_1, \dots, t_l$.

2.4. **The Yangian action.** Let $f = f(t_1,\dots,t_l)$ be a function. For a permutation $\sigma \in \mathbb{S}^l$, define the functions $[f]^{rat}_\sigma$ via the action of the simple transpositions $(i, i+1) \in \mathbb{S}^l$, $i = 1, \dots, l-1$, given by

$$[f]^{rat}_{(i,i+1)}(t_1,\dots,t_l) = f(t_1,\dots,t_{i+1},t_i,\dots,t_l)\, \frac{t_i - t_{i+1} - 1}{t_i - t_{i+1} + 1}.$$

Let $T_{ij}(u)$, $i,j = 1,2$, be the generating series for the Yangian $Y(gl(2))$ introduced in Section 2.2. Set

$$\widetilde{T}_{ij}(u) \,=\, T_{ij}(u) \prod_{a=1}^{n} \frac{u}{u - z_a - \lambda_a}\,, \qquad i,j = 1,2\,.$$

Here the rational function in the right hand side is understood as its Laurent series expansion at $u = \infty$. In this paper we always use this convention in formulas of this kind.

It is clear that the coefficients of the series $\widetilde{T}_{ij}(u)$ generate $Y(gl(2))$. Following [TV], define an action of the coefficients of the series $\widetilde{T}_{ij}(u)$ in the hypergeometric space $\mathfrak{F}(z,\lambda)$. Namely, for a function $f \in \mathfrak{F}_l(z,\lambda)$, set:

$$\begin{aligned}
(\widetilde{T}_{11}(u)f)(t_1,\dots,t_l) \;&=\; f(t_1,\dots,t_l) \prod_{a=1}^{n} \frac{u - z_a + \lambda_a}{u - z_a - \lambda_a} \prod_{a=1}^{l} \frac{u - t_a - 1}{u - t_a} \;+ \qquad (3)\\
&+\; \prod_{a=1}^{l} \frac{u - t_a - 1}{u - t_a} \sum_{a=1}^{l} \left[\frac{f(t_1,\dots,t_{l-1},u)}{u - t_l - 1} \prod_{b=1}^{n} \frac{t_l - z_b + \lambda_b}{t_l - z_b - \lambda_b} \right]^{rat}_{(a,l)},
\end{aligned}$$

$$\begin{aligned}
(\widetilde{T}_{22}(u)f)(t_1,\dots,t_l) \;&=\; f(t_1,\dots,t_l) \prod_{a=1}^{l} \frac{u - t_a + 1}{u - t_a} \;-\\
&-\; \prod_{a=1}^{l} \frac{u - t_a + 1}{u - t_a} \sum_{a=1}^{l} \left[\frac{f(u,t_2,\dots,t_l)}{u - t_1 + 1} \right]^{rat}_{(1,a)},
\end{aligned}$$

$$
\begin{aligned}
(\widetilde{T}_{12}(u)f)(t_1,\dots,t_{l+1}) \;&=\; \sum_{a=1}^{l+1}\Bigg[\frac{f(t_2,\dots,t_{l+1})}{u-t_1}\times\\
&\times\;\Bigg(\prod_{b=1}^{n}\frac{t_1-z_b+\lambda_b}{t_1-z_b-\lambda_b}\prod_{c=2}^{l+1}\frac{u-t_c+1}{u-t_c}\,\frac{t_1-t_c-1}{t_1-t_c+1}\,-\\
&-\;\prod_{b=1}^{n}\frac{u-z_b+\lambda_b}{u-z_b-\lambda_b}\prod_{c=2}^{l+1}\frac{u-t_c-1}{u-t_c}\Bigg)\Bigg]^{rat}_{(1,a)}-\\
-\prod_{a=1}^{l+1}\frac{u-t_a+1}{u-t_a}\sum_{\substack{a,b=1\\ a\neq b}}^{l+1}&\left[\frac{f(u,t_2,\dots,t_l)}{(u-t_1+1)\,(u-t_{l+1}+1)}\prod_{c=1}^{n}\frac{t_{l+1}-z_c+\lambda_c}{t_{l+1}-z_c-\lambda_c}\right]^{rat}_{(1,a)(b,l+1)}\,,
\end{aligned}
$$

$$
(\widetilde{T}_{21}(u)f)(t_1,\dots,t_{l-1}) \;=\; f(t_1,\dots,t_{l-1},u)\prod_{a=1}^{l-1}\frac{u-t_a-1}{u-t_a}\,,\qquad l>0\,,
$$

and set $\widetilde{T}_{21}(u)f=0$ if $f\in\mathfrak{F}_0(z,\lambda)$. Here $(1,a)$, (a,l), $(b,l+1)$ are transpositions.

By Lemma 4.2 in [TV], these formulas define a $Y(gl(2))$ module structure in the rational hypergeometric space $\mathfrak{F}(z,\lambda)$ for any $z,\lambda\in\mathbb{C}^n$.

2.5. **Bases of the rational hypergeometric space.** For natural numbers n,l, set $\mathcal{Z}^n_l=\{\bar{l}=(l_1,\dots,l_n)\in\mathbb{Z}^n_{\geqslant 0}\,|\,\sum_{a=1}^{n}l_a=l\}$. For $\bar{l}\in\mathcal{Z}^n_l$ and $i=0,1,\dots,n$, set $l^i=\sum_{a=1}^{i}l_a$.

For $\bar{l}\in\mathcal{Z}^n_l$, define the *rational weight function* $w_{\bar{l}}$ by

$$
w_{\bar{l}}(t,z,\lambda)=\sum_{\sigma\in\mathbb{S}^l}\left[\prod_{a=1}^{n}\frac{1}{l_a!}\prod_{b=l^{a-1}+1}^{l^a}\left(\frac{1}{t_b-z_a-\lambda_a}\prod_{c=1}^{a-1}\frac{t_b-z_c+\lambda_c}{t_b-z_c-\lambda_c}\right)\right]^{rat}_{\sigma}.
$$

Denote $V(z,\lambda)=V(z_1,\lambda_1)\otimes\ldots\otimes V(z_n,\lambda_n)$ the tensor product of evaluation Verma modules. The module $V(z,\lambda)$ has a basis given by monomials $\{f^{l_1}v_1\otimes\ldots\otimes f^{l_n}v_n\}$, where $l_i\in\mathbb{Z}_{\geqslant 0}$ and v_i are highest vectors of $V(z_i,\lambda_i)$. The dual space $V^*(z,\lambda)$ has the dual basis denoted by $(f^{l_1}v_1\otimes\ldots\otimes f^{l_n}v_n)^*$. Define a $\mathbb{C}$-linear map $\nu(z,\lambda)\,:\,V^*(z,\lambda)\,\to\,\mathfrak{F}(z,\lambda)$ by the formula

$$
\nu(z,\lambda)\,:\,(f^{l_1}v_1\otimes\ldots\otimes f^{l_n}v_n)^*\mapsto w_{\bar{l}}(t,z,\lambda). \tag{4}
$$

Lemma 1. *(Cf. Lemma 4.5, Theorem 4.7 in* [TV]*.) The map $\nu(z,\lambda)$ is a homomorphism of Yangian modules. Moreover, if $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant 0}$ for all $a>b$, $a,b=1,\dots,n$, then the map $\nu(z,\lambda)$ is an isomorphism.* □

For $z,\lambda\in\mathbb{C}^n$ and permutations $\sigma,\sigma'\in\mathbb{S}^n$, define $z^{\sigma'}_{\sigma},\lambda^{\sigma'}_{\sigma}\in\mathbb{C}^n$ by equations $(z^{\sigma'}_{\sigma})_a+(\lambda^{\sigma'}_{\sigma})_a=z_{\sigma(a)}+\lambda_{\sigma(a)}$ and $(z^{\sigma'}_{\sigma})_a-(\lambda^{\sigma'}_{\sigma})_a=z_{\sigma'(a)}-\lambda_{\sigma'(a)}$, $a=1,\dots,n$.

Lemma 2. *For any permutations $\sigma,\sigma'\in\mathbb{S}^n$, the identity map* $\mathrm{Id}:\mathfrak{F}(z^{\sigma'}_{\sigma},\lambda^{\sigma'}_{\sigma})\to\mathfrak{F}(z,\lambda)$ *is an isomorphism of Yangian modules.*

Proof: The characteristic property of $z^{\sigma'}_{\sigma},\lambda^{\sigma'}_{\sigma}$ is

$$
\prod_{a=1}^{n}(u-(z^{\sigma'}_{\sigma})_a-(\lambda^{\sigma'}_{\sigma})_a)=\prod_{a=1}^{n}(u-z_a-\lambda_a),\qquad \prod_{a=1}^{n}(u-(z^{\sigma'}_{\sigma})_a+(\lambda^{\sigma'}_{\sigma})_a)=\prod_{a=1}^{n}(u-z_a+\lambda_a).
$$

The definition of the space $\mathfrak{F}(z,\lambda)$ and of the Yangian action depends on z,λ only through polynomials $\prod_{a=1}^{n}(u-z_a+\lambda_a)$ and $\prod_{a=1}^{n}(u-z_a-\lambda_a)$, see (3). $\square$

Lemma 3. *Let* $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant 0}$, $a,b=1,\dots,n$. *Then for any permutations* $\sigma,\sigma'\in\mathbb{S}^n$, *the Yangian modules* $V(z,\lambda)$ *and* $V(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$ *are isomorphic.*

Proof 1: Under the assumption of the Lemma, the Shapovalov map $Sh:\ V(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})\to V^*(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$ is an isomorpism of Yangian modules for any permutations $\sigma,\sigma'\in\mathbb{S}^n$. Lemma 3 follows from Lemmas 1 and 2. $\square$

Proof 2: Both modules $V(z,\lambda)$ and $V(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$ are irreducible Yangian modules of highest weight

$$\prod_{a=1}^{n}\frac{u-z_a+\lambda_a}{u-z_a-\lambda_a}\,.$$

For the definition of highest weight and the classification of irreducible Yangian modules, see [T]. $\square$

Let z,λ be as in Lemma 3. Let $(\sigma'\otimes\sigma)(z,\lambda):V(z,\lambda)\to V(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$ be the isomorphism of Yangian modules such that

$$(\sigma'\otimes\sigma)(z,\lambda):v_1\otimes\ldots\otimes v_n\mapsto v_1'\otimes\ldots\otimes v_n',$$

where v_i,v_i' are highest vectors generating $V(z_i,\lambda_i),V((z_\sigma^{\sigma'})_i,(\lambda_\sigma^{\sigma'})_i)$.

We have

$$(\sigma'\otimes\sigma)(z,\lambda)=(Sh(\lambda_\sigma^{\sigma'}))^{-1}\circ(\nu(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}))^{-1}\circ\nu(z,\lambda)\circ Sh(\lambda),\tag{5}$$

where the map ν is given by (4).

Example 4. Let $n=2$. Consider the transpositions $\sigma=\sigma'=(1,2)$. Then

$$z_{(1,2)}^{(1,2)}=(z_2,z_1),\qquad \lambda_{(1,2)}^{(1,2)}=(\lambda_2,\lambda_1).\tag{6}$$

The isomorpism $\widehat{R}(z,\lambda):V(z,\lambda)\to V(z_{(1,2)}^{(1,2)},\lambda_{(1,2)}^{(1,2)})$, given by formula (5), will be called *the rational* $\widehat{R}$ *matrix.*

Example 5. Let $n=2$. Consider the case $\sigma=id$, $\sigma'=(1,2)$. Then

$$\begin{aligned}(z_{id}^{(1,2)})_1&=\frac12(z_1+z_2+\lambda_1-\lambda_2), & (z_{id}^{(1,2)})_2&=\frac12(z_1+z_2-\lambda_1+\lambda_2),\\ (\lambda_{id}^{(1,2)})_1&=\frac12(z_1-z_2+\lambda_1+\lambda_2), & (\lambda_{id}^{(1,2)})_2&=\frac12(-z_1+z_2+\lambda_1+\lambda_2).\end{aligned}\tag{7}$$

We call the isomorphism $N(z,\lambda):V(z,\lambda)\to V(z_{id}^{(1,2)},\lambda_{id}^{(1,2)})$, given by formula (5), *the rational* N *matrix.*

Example 6. Let $n=2$. Consider the case $\sigma'=id$, $\sigma=(1,2)$. Then

$$(z_{(1,2)}^{id})_1=\frac{1}{2}(z_1+z_2-\lambda_1+\lambda_2), \qquad (z_{(1,2)}^{id})_2=\frac{1}{2}(z_1+z_2+\lambda_1-\lambda_2),$$
$$(\lambda_{(1,2)}^{id})_1=\frac{1}{2}(-z_1+z_2+\lambda_1+\lambda_2), \qquad (\lambda_{(1,2)}^{id})_2=\frac{1}{2}(z_1-z_2+\lambda_1+\lambda_2). \tag{8}$$

We call the isomorphism $D(z,\lambda):V(z,\lambda)\to V(z_{(1,2)}^{id},\lambda_{(1,2)}^{id})$, given by formula (5), *the rational D matrix*.

Our notations of N and D matrices come from the words "numerator" and "denominator". We describe properties of the $\widehat{R}$, D and N matrices in the next Section.

2.6. **Properties of R, D and N matrices.** Let n be a natural number. Let $z,\lambda\in\mathbb{C}^n$ be such that $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant 0}$, for all $a,b=1,\dots,n$.

Lemma 7. *The group $\mathbb{S}^n\times\mathbb{S}^n$ transitively acts on the family of isomorphic Yangian modules $\{V(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}),\sigma,\sigma'\in\mathbb{S}^n\}$, $(\tau'\times\tau)(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}):V(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})\to V(z_{\tau\sigma}^{\tau'\sigma'},\lambda_{\tau\sigma}^{\tau'\sigma'})$. The action of generators is given by*

$$(a,a+1)\times id=N_{a,a+1}, \qquad id\times(a,a+1)=D_{a,a+1},$$

where $N_{a,a+1}$, $D_{a,a+1}$ are N and D matrices acting in the ath and $(a+1)$st factors of the the corresponding tensor product. We also have

$$N_{a,a+1}(z_{(a,a+1)}^{id},\lambda_{(a,a+1)}^{id})D_{a,a+1}(z,\lambda)=D_{a,a+1}(z_{id}^{(a,a+1)},\lambda_{id}^{(a,a+1)})N_{a,a+1}(z,\lambda)=\widehat{R}_{a,a+1}(z,\lambda).$$

Proof: The Lemma follows from the definitions of $\widehat{R}$, D and N matrices. $\square$

Let $n=2$. Consider the $sl(2)$ Verma modules $V(\lambda_1),V(\lambda_2)$ of highest weights λ_1,λ_2 with highest vectors v_1,v_2. We have a decomposition of $sl(2)$ modules

$$V(\lambda_1)\otimes V(\lambda_2)=\bigoplus_{l=0}^{\infty}V(\lambda_1+\lambda_2-l).$$

We choose singular vectors $v_{(\lambda_1,\lambda_2;\,l)}\in V(\lambda_1)\otimes V(\lambda_2)$ generating $V(\lambda_1+\lambda_2-l)$, so that in the standard basis the coefficient of $f^lv_1\otimes v_2$ is $(B_l(\lambda_1))^{-1}$:

$$ev_{(\lambda_1,\lambda_2;\,l)}=0, \quad v_{(\lambda_1,\lambda_2;\,l)}=(B_l(\lambda_1))^{-1}\,f^lv_1\otimes v_2+\sum_{a=1}^{l}c_a^l(\lambda)f^{l-a}v_1\otimes f^av_2, \quad l=0,1,\dots,$$

where $B_l(\lambda_1)=B_\lambda(f^lv,f^lv)$ is the value of the Shapovalov form, see (1), and $c_a^l(\lambda)$ are complex numbers.

Theorem 8. *(Spectral decomposition.) Let $A:V(z,\lambda)\to V(u,\omega)$ be either $\widehat{R}$, D or N matrix, where $u,\omega\in\mathbb{C}^2$ are the parameters of the target module. Then A commutes with the $sl(2)$ action and maps*

$$A\,:\,v_{(\lambda_1,\lambda_2;\,l)}\;\mapsto\;\frac{B_l((\omega_{id}^{(1,2)})_1)}{B_l((\lambda_{id}^{(1,2)})_1)}\,v_{(\omega_1,\omega_2;\,l)},$$

where $B_l(\lambda)=B_\lambda(f^lv,f^lv)$ is the value of the Shapovalov form, see (1).

Proof: In the case of $\widehat{R}$ matrix the spectral decomposition is well known, see Proposition 12.5.4 in [CP] and formula (3.5) in [TV]. In the two other cases the proof is similar. $\square$

Let n be any natural number, $z,\lambda \in \mathbb{C}^n$. We say that the parameters $z,\lambda \in \mathbb{C}^n$ are *in a rational resonance* if the Yangian module $V(z,\lambda)$ is reducible. We say that the parameters z,λ are *in the first rational resonance* if there exists a unique pair of indices $a,b \in \{1,\dots,n\}$ such that $z_a - z_b + \lambda_a + \lambda_b = k \in \mathbb{Z}_{\geqslant 0}$ and if $a \leqslant b$. We say that the parameters z,λ are *in the second rational resonance* if there exists a unique pair of indices $a,b \in \{1,\dots,n\}$ such that $z_a - z_b + \lambda_a + \lambda_b = k \in \mathbb{Z}_{\geqslant 0}$ and if $a > b$.

Let z,λ be in the first rational resonance. Then the Yangian module $V(z,\lambda)$ has a unique nontrivial submodule. The submodule is isomorphic to a tensor product $V(z',\lambda')$ of evaluation Verma modules with parameters $z',\lambda' \in \mathbb{C}^n$ such that $(z',\lambda') \neq (z,\lambda)$. For example, if $2\lambda_1 = k \in \mathbb{Z}_{\geqslant 0}$, i.e. $a=b=1$, then $z'=z$, $\lambda' = (-\lambda_1 - 1, \lambda_2, \dots, \lambda_n)$.

Let z,λ be in the second rational resonance. Then the Yangian module $V(z,\lambda)$ has a unique nontrivial submodule. The factor module is isomorphic to a tensor product $V(z',\lambda')$ of evaluation Verma modules with parameters $z',\lambda' \in \mathbb{C}^n$ such that $(z',\lambda') \neq (z,\lambda)$. For example, if $z_2 - z_1 + \lambda_1 + \lambda_2 = k \in \mathbb{Z}_{\geqslant 0}$, i.e. $a=2$, $b=1$, then $z' = z^{(12)}_{id}$, $\lambda' = (-(\lambda^{(12)}_{id})_1 - 1, (\lambda^{(12)}_{id})_2, \dots, (\lambda^{(12)}_{id})_n)$, see Section 12.1 in [CP] and references therein.

Lemma 9. *Let $z,\lambda \in \mathbb{C}^n$ and $z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma} \in \mathbb{C}^n$ be either both in the first rational resonance or both in the second rational resonance. Then the map $(\sigma' \times \sigma)(z,\lambda) : V(z,\lambda) \to V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$ is a well defined isomorphism of Yangian modules. It maps the submodule $V(z',\lambda') \subset V(z,\lambda)$ onto the submodule $V((z^{\sigma'}_{\sigma})', (\lambda^{\sigma'}_{\sigma})') \subset V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$, and the map $(\sigma' \times \sigma)(z,\lambda)$ restricted to the submodule $V(z',\lambda')$ coincides with the map $(\sigma'\times\sigma)(z',\lambda')$ up to a non-zero scalar multiplier, depending on the choice of the inclusions $V(z',\lambda') \hookrightarrow V(z,\lambda)$ and $V((z^{\sigma'}_{\sigma})', (\lambda^{\sigma'}_{\sigma})') \hookrightarrow V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$.*

Proof: Lemma 9 follows from Theorem 8. $\square$

Lemma 10. *Let $z,\lambda \in \mathbb{C}^n$ be in the first rational resonance and $z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma} \in \mathbb{C}^n$ be in the second rational resonance. Then the map $(\sigma' \times \sigma)(z,\lambda) : V(z,\lambda) \to V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$ is a well defined homomorphism of Yangian modules. The kernel of this homomorphism is the submodule $V(z',\lambda') \subset V(z,\lambda)$ and the image is the proper submodule in $V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$.*

Proof: Lemma 10 follows from Theorem 8. $\square$

Lemma 11. *Let $z,\lambda \in \mathbb{C}^n$ be in the second rational resonance and $z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma} \in \mathbb{C}^n$ be in the first rational resonance. Then the map $(\sigma' \times \sigma)(\tilde z,\tilde\lambda) : V(\tilde z,\tilde\lambda) \to V(\tilde z^{\sigma'}_{\sigma}, \tilde\lambda^{\sigma'}_{\sigma})$ has a simple pole at $\tilde z = z, \tilde\lambda = \lambda$. The residue $Res := \mathrm{res}_{\tilde z = z,\, \tilde\lambda = \lambda} (\sigma' \times \sigma)(\tilde z,\tilde\lambda)$ is a well defined homomorphism of Yangian modules. The kernel of this homomorphism is the nontrivial submodule U of $V(z,\lambda)$ and the image is the submodule $V((z^{\sigma'}_{\sigma})', (\lambda^{\sigma'}_{\sigma})') \subset V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$. Thus, up to a scalar multiplier, depending on the choice of the factor map $V(z,\lambda) \to V(z',\lambda') \simeq V(z,\lambda)/U$, and the inclusion $V((z^{\sigma'}_{\sigma})', (\lambda^{\sigma'}_{\sigma})') \hookrightarrow V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$, the map Res defines a homomorphism $V(z',\lambda') \to V((z^{\sigma'}_{\sigma})', (\lambda^{\sigma'}_{\sigma})')$. The scalar multiplier can be chosen so that the map Res coincides with the isomorphism $(\sigma' \times \sigma)(z',\lambda')$.*

Proof: Lemma 11 follows from Theorem 8. $\square$

2.7. **The first resonance in the rational hypergeometric space.** Let $z, \lambda \in \mathbb{C}^n$. Suppose z, λ are in the first rational resonance. We have $z_a - z_b + \lambda_a + \lambda_b = k$ for some $k \in \mathbb{Z}_{\geqslant 0}$, $a, b \in \{1, \ldots, n\}$, $a \leqslant b$. Choose $\sigma, \sigma' \in \mathbb{S}^n$, such that $\sigma(a) = \sigma'(b) = 1$. Then by Lemma 9, the map $\sigma' \times \sigma$ is a well defined isomorphism and $2(\lambda_\sigma^{\sigma'})_1 = k$.

Let $z, \lambda \in \mathbb{C}^n$ and $2\lambda_1 = k$, $k \in \mathbb{Z}_{\geqslant 0}$. Set $z' = z$, $\lambda' = (-\lambda_1 - 1, \lambda_2, \ldots, \lambda_n)$.

Define a linear map $\iota^*(z, \lambda) : \mathfrak{F}(z, \lambda) \to \mathfrak{F}(z', \lambda')$ as follows. For a function $f(t_1, \ldots, t_l) \in \mathfrak{F}_l(z, \lambda)$, set

$$\tilde{f}(t_1, \ldots, t_l) = f(t_1, \ldots, t_l)(t_1 - z_1 - \lambda_1) \prod_{a=1}^{k+1} \prod_{b=k+2}^{l} \frac{t_a - t_b + 1}{t_a - t_b - 1}, \qquad \text{if } l > k,$$

and $\tilde{f} = 0$, if $l \leqslant k$. Define

$$(\iota^* f)(t_1 \ldots, t_{l-k-1}) = \tilde{f}(z_1 + \lambda_1, z_1 + \lambda_1 - 1, \ldots, z_1 - \lambda_1, t_1, \ldots, t_{l-k-1}).$$

Theorem 12. *The map $\iota^*(z, \lambda) : \mathfrak{F}(z, \lambda) \to \mathfrak{F}(z', \lambda')$ is a surjective homomorphism of Yangian modules.*

Proof: The map ι^* is well defined, it is a surjection, see the definition of the rational hypergeometric space.

If $t_a = z_1 + \lambda_1 - a + 1$, $a = 1, \ldots, k+1$, then we have the following identities:

$$\prod_{a=1}^{k+1} \frac{u - t_a + 1}{u - t_a} \prod_{a=1}^{n} (u - z_a - \lambda_a) = \prod_{a=1}^{n} (u - z'_a - \lambda'_a),$$

$$\prod_{a=1}^{k+1} \frac{t_a - u + 1}{t_a - u - 1} \prod_{a=1}^{n} \frac{u - z_a + \lambda_a}{t - z_a - \lambda_a} = \prod_{a=1}^{n} \frac{u - z'_a + \lambda'_a}{u - z'_a - \lambda'_a}.$$

It is a straightforward calculation to check that the map ι^* commutes with the Yangian action given by (3), using the above identities. $\square$

Let $z, \lambda \in \mathbb{C}^n$ and $2\lambda_1 = k \in \mathbb{Z}_{\geqslant 0}$. Assume $z_a - z_b + \lambda_a + \lambda_b \notin \mathbb{Z}_{\geqslant 0}$ for all $a > b$, $a, b = 1, \ldots, n$. Then by Lemma 1,

$$\mathfrak{F}(z, \lambda) \simeq V^*(z, \lambda) \simeq V^*(z_1, \lambda_1)) \otimes V^*((z_2, \ldots, z_n), (\lambda_2, \ldots, \lambda_n)),$$

$$\mathfrak{F}(z', \lambda') \simeq V^*(z', \lambda') \simeq V^*(z'_1, \lambda'_1) \otimes V^*((z_2, \ldots, z_n), (\lambda_2, \ldots, \lambda_n)).$$

The Yangian module $V^*(z_1, \lambda_1)$ has a submodule, isomorphic to $(k+1)$-dimensional irreducible evaluation module $L(z_1, \lambda_1)$. Then $\iota^*(z, \lambda) = \iota_1^* \otimes \mathrm{Id}$, where

$$\iota_1^* : V^*(z_1, \lambda_1) \to V^*(z_1, \lambda_1)/L(z_1, \lambda_1) \simeq V^*(z'_1, \lambda'_1)$$

is the factorization morphism mapping $f^{k+1} v_1 \otimes v_2 \otimes \ldots \otimes v_n$ to $D(k) v'_1 \otimes \ldots \otimes v'_n$. The constant $D(k)$ is given by

$$D(k) = \prod_{a=2}^{k+1} \frac{1}{t_a - z_1 - \lambda_1} \prod_{1 \leqslant a < b \leqslant k+1} \frac{t_a - t_b}{t_a - t_b + 1},$$

where we set $t_a = z_1 + \lambda_1 - a + 1$, $a = 1, \ldots, k+1$.

3. Trigonometric hypergeometric space of functions

3.1. The quantum group $U_q sl(2)$. Let q be a complex number different from ± 1. Let p be a complex number such that $q = \exp(\pi i/p)$. We always assume $\mathrm{Re}\, p < 0$. For a complex number x, by q^x we mean $\exp(\pi i x/p)$.

Let e_q, f_q, q^h, q^{-h} be generators of $U_q sl(2)$ such that

$$q^h q^{-h} = q^{-h} q^h = 1, \qquad [e_q, f_q] = \frac{q^{2h} - q^{-2h}}{q - q^{-1}}, \qquad q^h e_q = q e_q q^h, \qquad q^h f_q = q^{-1} f_q q^h.$$

A comultiplication $\Delta_q : U_q sl(2) \to U_q sl(2) \otimes U_q sl(2)$ is given by

$$\Delta_q(q^{\pm h)} = q^{\pm h} \otimes q^{\pm h}, \qquad \Delta_q(e_q) = e_q \otimes q^h + q^{-h} \otimes e_q, \qquad \Delta_q(f_q) = f_q \otimes q^h + q^{-h} \otimes f_q.$$

The comultiplication defines a module structure on tensor products of $U_q sl(2)$ modules.

An antipode $S_q : U_q sl(2) \to U_q sl(2)$ is given by

$$S_q(e_q) = f_q, \qquad S_q(f_q) = e_q, \qquad S_q(q^{\pm h}) = q^{\pm h}.$$

The antipode defines a module structure on a space dual to an $U_q sl(2)$ module.

For $\lambda \in \mathbb{C}$, denote $V_q(\lambda)$ the $U_q sl(2)$ Verma module with highest weight q^λ and highest vector v^q. The vectors $\{f_q^l v^q,\ l \in \mathbb{Z}_{\geqslant 0}\}$ form a basis in $V_q(\lambda)$. The quantum Shapovalov form on $V_q(\lambda)$ is a bilinear form B_λ^q such that

$$B_\lambda^q\,(f_q^l v^q, f_q^l v^q) = [l]_q!\,[2\lambda - a]_q, \qquad B_\lambda^q\,(f_q^l v^q, f_q^k v^q) = 0,\ l \neq k, \tag{9}$$

where $[n]_q = (q^n - q^{-n})/(q - q^{-1}) = \sin(\pi n/p)/\sin(\pi/p)$ are the q-numbers, $[l]_q! = [1]_q![2]_q! \dots [l]_q!$.

The quantum Shapovalov form B^q on a tensor product $V_q(\lambda_1) \otimes \dots \otimes V_q(\lambda_n)$ is defined by $B^q = B_{\lambda_1}^q \otimes \dots \otimes B_{\lambda_n}^q$. The dual map to the quantum Shapovalov form defines a map of $U_q sl(2)$ modules

$$Sh_q :\ V_q(\lambda_1) \otimes \dots \otimes V_q(\lambda_n) \to (V_q(\lambda_1) \otimes \dots \otimes V_q(\lambda_n))^*\,.$$

For $\lambda \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$, denote $L_q(\lambda)$ the q deformation of $sl(2)$ module $L(\lambda)$. $L_q(\lambda)$ is a $(2\lambda + 1)$-dimensional $U_q sl(2)$ module.

3.2. The Hopf algebra $U_q\widehat{gl(2)}$. The quantum affine algebra $U_q\widehat{gl(2)}$ is a unital associative algebra with generators $L_{ij}^{(+0)}$, $L_{ji}^{(-0)}$, $1 \leqslant j \leqslant i \leqslant 2$, and $L_{ij}^{(s)}$, $i, j = 1, 2$, $s = \pm 1, \pm 2, \dots$, subject to relations (10).

Let $e_{ij} \in \mathrm{End}(\mathbb{C}^2)$ $i, j = 1, 2$, be the standard matrix units with the only nonzero entry 1 at the intersection of the i-th row and j-th column. Set

$$\begin{aligned} R(\xi) \;&=\; (\xi q - q^{-1})\,(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) \;+ \\ &+\; (\xi - 1)\,(e_{12} \otimes e_{12} + e_{21} \otimes e_{21}) \,+\, \xi(q - q^{-1})\, e_{12} \otimes e_{21} \,+\, (q - q^{-1})\, e_{21} \otimes e_{12}\,. \end{aligned}$$

Introduce the generating series $L_{ij}^{\pm}(u) = L_{ij}^{(\pm 0)} + \sum_{s=1}^{\infty} L_{ij}^{(\pm s)} u^{\pm s}$. The relations in $U_q\widehat{gl(2)}$ have the form

$$\begin{gathered} R(\xi/\zeta)\, L_{(1)}^{\pm}(\xi)\, L_{(2)}^{\pm}(\zeta) \;=\; L_{(2)}^{\pm}(\zeta)\, L_{(1)}^{\pm}(\xi)\, R(\xi/\zeta)\,, \\ R(\xi/\zeta)\, L_{(1)}^{+}(\xi)\, L_{(2)}^{-}(\zeta) \;=\; L_{(2)}^{-}(\zeta)\, L_{(1)}^{+}(\xi)\, R(\xi/\zeta)\,, \\ L_{11}^{(\pm 0)} L_{22}^{(\pm 0)} \;=\; 1\,, \qquad L_{22}^{(\pm 0)} L_{11}^{(\pm 0)} \;=\; 1\,, \qquad L_{ii}^{(\pm 0)} L_{ii}^{(\mp 0)} \;=\; 1\,, \qquad i = 1, 2\,, \end{gathered} \tag{10}$$

where $L^{\pm}_{(1)}(\xi) = L^{\pm}(\xi) \otimes 1$ and $L^{\pm}_{(2)}(\xi) = 1 \otimes L^{\pm}(\xi)$.

In this paper we take tensor products and dual of $U_q\widehat{gl(2)}$ modules using the comultiplication $\widehat{\Delta}_q \,:\, U_q\widehat{gl(2)} \to U_q\widehat{gl(2)} \otimes U_q\widehat{gl(2)}$ and the antipode $\widehat{S}_q \,:\, U_q\widehat{gl(2)} \to U_q\widehat{gl(2)}$ given by

$$\widehat{\Delta}_q \,:\, L^{\pm}_{ij}(\xi) \mapsto \sum_{a=1}^{2} L^{\pm}_{ai}(\xi) \otimes L^{\pm}_{ja}(\xi), \tag{11}$$

$$\widehat{S}_q \,:\, L^{\pm}_{ij}(\xi) \mapsto L^{\pm}_{ji}(\xi), \qquad i,j = 1,2.$$

Note that our choice of comultiplication is, in a sense, opposite to the comultiplication we chose for Yangian, see (2).

For a complex number ζ, there is an automorphism $\rho^q_\zeta \,:\, U_q\widehat{gl(2)} \to U_q\widehat{gl(2)}$ of the form

$$\rho^q_\zeta \,:\, L^{\pm}_{ij}(\zeta) \mapsto L^{\pm}_{ij}(\xi/\zeta), \qquad i,j = 1,2.$$

The evaluation morphism $\epsilon^q \,:\, U_q\widehat{gl(2)} \to U_q sl(2)$ has the form

$$\epsilon^q \,:\, L^{\pm}_{11}(\xi) \,\mapsto\, q^{\mp h} - q^{\pm h}\xi^{\pm 1}\,, \qquad \epsilon^q \,:\, L^{\pm}_{22}(\xi) \,\mapsto\, q^{\pm h} - q^{\mp h}\xi^{\pm 1}\,,$$

$$\epsilon^q \,:\, L^{+}_{12}(\xi) \,\mapsto\, -f_q\,(q - q^{-1})\,\xi\,, \qquad \epsilon^q \,:\, L^{-}_{12}(\xi) \,\mapsto\, f_q\,(q - q^{-1}),$$

$$\epsilon^q \,:\, L^{+}_{21}(\xi) \,\mapsto\, -e_q\,(q - q^{-1})\,, \qquad \epsilon^q \,:\, L^{-}_{21}(\xi) \,\mapsto\, e_q\,(q - q^{-1})\,\xi^{-1}\,.$$

For complex numbers z, λ, denote $V_q(z,\lambda)$ the $U_q sl(2)$ Verma module $V_q(\lambda)$ endowed with a structure of $U_q\widehat{gl(2)}$ module via pull back with respect to the map $\epsilon^q \circ \rho^q_{q^{2z}}$. The module $V_q(z,\lambda)$ is called *the quantum evaluation Verma module*.

For $\lambda \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$, denote $L_q(z,\lambda)$ the $U_q sl(2)$ module $L_q(\lambda)$ endowed with a structure of $U_q\widehat{gl(2)}$ module via pull back with respect to the map $\epsilon^q \circ \rho^q_{q^{2z}}$. The module $L_q(z,\lambda)$ is called *the quantum evaluation finite dimensional module*.

For more detail on the affine quantum group $U_q\widehat{gl(2)}$ see [CP], [TV].

3.3. **The trigonometric hypergeometric space.** Fix a natural number n, $z = (z_1, \dots, z_n)$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and a complex number $q = \exp(\pi i/p)$.

Define *a trigonometric hypergeometric space* $\mathfrak{F}^{\,q}(z,\lambda) = \bigoplus_{l=0}^{\infty} \mathfrak{F}^{\,q}_{\,l}(z,\lambda)$, where $\mathfrak{F}^{\,q}_{\,l}(z,\lambda)$ is the space of functions of the form

$$P(q^{2t_1}, \dots, q^{2t_n}) \prod_{a=1}^{n} \prod_{b=1}^{l} \frac{q^{z_a - t_b}}{\sin(\pi(t_b - z_a - \lambda_a)/p)} \prod_{1 \leqslant a < b \leqslant l} \frac{\sin(\pi(t_a - t_b)/p)}{\sin(\pi(t_a - t_b + 1)/p)}\,.$$

Here P is a polynomial with complex coefficients which is symmetric in variables $q^{2t_1}, \dots, q^{2t_n}$ and has degree less than n in each of variables $q^{2t_1}, \dots, q^{2t_n}$.

3.4. **The $U_q\widehat{gl(2)}$ action.** Let $f = f(t_1, \dots, t_l)$ be a function. For a permutation $\sigma \in \mathbb{S}^l$, define the functions $[f]^{trig}_\sigma$ via the action of the simple transpositions $(i, i+1) \in \mathbb{S}^l$, $i = 1, \dots, l-1$, given by

$$[f]^{trig}_{(i,i+1)}(t_1, \dots, t_l) = f(t_1, \dots, t_{i+1}, t_i, \dots, t_l)\, \frac{\sin(\pi(t_i - t_{i+1} - 1)/p)}{\sin(\pi(t_i - t_{i+1} + 1)/p)}.$$

Let $L^{\pm}_{ij}(u)$, $i,j=1,2$, be the generating series for the quantum affine group $U_q\widehat{gl(2)}$ introduced in Section 3.2. Set

$$\widetilde{L}^{\pm}_{ij}(\xi) \,=\, L^{\pm}_{ij}(\xi))\prod_{a=1}^{n}\frac{\pm i q^{\pm(z_a-u)}}{2\sin(\pi(u-z_a-\lambda_a)/p)}\,, \qquad i,j=1,2\,,$$

where $\xi = q^{2u}$. It is clear that the coefficients of the series $\widetilde{L}^{\pm}_{ij}(u)$ generate $U_q\widehat{gl(2)}$. Following [TV], define an action of the coefficients of the series $\widetilde{L}^{\pm}_{ij}(u)$ in the hypergemetric space $\mathfrak{F}^{\,q}(z,\lambda)$. Namely, for a function $f\in\mathfrak{F}^{\,q}_{\,l}(z,\lambda)$, set:

$$(\widetilde{L}^{\pm}_{11}(\xi)f)(t_1,\dots,t_l) = \tag{12}$$
$$= f(t_1,\dots,t_l)\prod_{a=1}^{n}\frac{\sin(\pi(u-z_a+\lambda_a)/p)}{\sin(\pi(u-z_a-\lambda_a)/p)}\prod_{a=1}^{l}\frac{\sin(\pi(u-t_a-1)/p)}{\sin(\pi(u-t_a)/p)} \;+\sin(\pi/p)\times$$
$$\times\prod_{a=1}^{l}\frac{\sin(\pi(u-t_a-1)/p)}{\sin(\pi(u-t_a)/p)}\sum_{a=1}^{l}\left[\frac{f(t_1,\dots,t_{l-1},u)\,q^{u-t_l}}{\sin(\pi(u-t_l-1)/p)}\prod_{b=1}^{n}\frac{\sin(\pi(t_l-z_b+\lambda_b)/p)}{\sin(\pi(t_l-z_b-\lambda_b)/p)}\right]^{trig}_{(a,l)},$$

$$(\widetilde{L}^{\pm}_{22}(\xi)f)(t_1,\dots,t_l) = f(t_1,\dots,t_l)\prod_{a=1}^{l}\frac{\sin(\pi(u-t_a+1)/p)}{\sin(\pi(u-t_a)/p)} -$$
$$-\,\sin(\pi/p)\prod_{a=1}^{l}\frac{\sin(\pi(u-t_a+1)/p)}{\sin(\pi(u-t_a)/p)}\sum_{a=1}^{l}\left[\frac{f(u,t_2,\dots,t_l)q^{u-t_1}}{\sin(\pi(u-t_1+1)/p)}\right]^{trig}_{(1,a)},$$

$$(\widetilde{L}_{12}(\xi)f)(t_1,\dots,t_{l+1}) \,=\, \sin(\pi/p)\sum_{a=1}^{l+1}\left[\frac{f(t_2,\dots,t_{l+1})\,q^{u-t_1}}{\sin(\pi(u-t_1)/p)}\times\right.$$
$$\times\left(\prod_{b=1}^{n}\frac{\sin(\pi(t_1-z_b+\lambda_b)/p)}{\sin(\pi(t_1-z_b-\lambda_b)/p)}\prod_{c=2}^{l+1}\frac{\sin(\pi(u-t_c+1)/p)}{\sin(\pi(u-t_c)/p)}\frac{\sin(\pi(t_1-t_c-1)/p)}{\sin(\pi(t_1-t_c+1)/p)} -\right.$$
$$\left.\left.-\prod_{b=1}^{n}\frac{\sin(\pi(u-z_b+\lambda_b)/p)}{\sin(\pi(u-z_b-\lambda_b)/p)}\prod_{c=2}^{l+1}\frac{\sin(\pi(u-t_c-1)/p)}{\sin(\pi(u-t_c)/p)}\right)\right]^{trig}_{(1,a)} -$$
$$-\;\sin^2(\pi/p)\prod_{a=1}^{l+1}\frac{\sin(\pi(u-t_a+1)/p)}{\sin(\pi(u-t_a)/p)}\sum_{a,b=1,\;a\neq b}^{l+1}\left[f(u,t_2,\dots,t_l)\times\right.$$
$$\left.\times\;\frac{q^{2u-t_1-t_{l+1}}}{\sin(\pi(u-t_1+1)/p)\,\sin(\pi(u-t_{l+1}+1)/p)}\prod_{c=1}^{n}\frac{\sin(\pi(t_{l+1}-z_c+\lambda_c)/p)}{\sin(\pi(t_{l+1}-z_c-\lambda_c)/p)}\right]^{trig}_{(1,a)(b,l+1)},$$

$$(\widetilde{L}^{\pm}_{21}(\xi)f)(t_1,\dots,t_{l-1}) \,=\, f(t_1,\dots,t_{l-1},u)\prod_{a=1}^{l-1}\frac{\sin(\pi(u-t_a-1)/p)}{\sin(\pi(u-t_a)/p)}\,, \qquad l>0\,,$$

and set $\widetilde{L}^{\pm}_{21}(\xi)f \,=\, 0$ if $f\,\in\,\mathfrak{F}^{\,q}_{\,0}(z,\lambda)$. Here $(1,a)$, (a,l), $(b,l+1)$ are transpositions, $\xi=q^{2u}$.

By Lemma 4.15 in [TV], these formulas define a $U_q\widehat{gl(2)}$ module structure in the trigonometric hypergeometric space $\mathfrak{F}^{\,q}(z,\lambda)$ for any $z,\lambda\in\mathbb{C}^n$.

3.5. **Bases of the trigonometric hypergeometric space.** For $\bar{l} \in \mathcal{Z}_l^n$, define the *trigonometric weight function* $W_{\bar{l}}$ by

$$W_{\bar{l}}(t,z,\lambda) = \sum_{\sigma\in\mathbb{S}^l}\left[\prod_{a=1}^{n}\prod_{d=1}^{l_m}\frac{\sin(\pi/p)}{\sin(\pi d/p)}\prod_{b=l^{a-1}+1}^{l^a}\frac{\exp(\pi i(z_a-t_b)/p)}{\sin(\pi(t_b-z_a-\lambda_a)/p)}\prod_{c=1}^{a-1}\frac{\sin(\pi(t_b-z_c+\lambda_c)/p)}{\sin(\pi(t_b-z_c-\lambda_c)/p)}\right]_{\sigma}^{trig}.$$

Denote $V_q(z,\lambda) = V_q(z_1,\lambda_1)\otimes\ldots\otimes V_q(z_n,\lambda_n)$ the tensor product of evaluation Verma modules. The module $V_q(z,\lambda)$ has a basis given by monomials $\{f_q^{l_1}v_1^q\otimes\ldots\otimes f_q^{l_n}v_n^q\}$, where $l_i\in\mathbb{Z}_{\geqslant 0}$ and v_i^q are highest vectors of $V_q(z_i,\lambda_i)$. The dual space $V_q^*(z,\lambda)$ has the dual basis denoted by $(f_q^{l_1}v_1^q\otimes\ldots\otimes f_q^{l_n}v_n^q)^*$. Define a $\mathbb{C}$-linear map $\nu_q(z,\lambda)\,:\,V_q^*(z,\lambda)\to\mathfrak{F}^{\,q}(z,\lambda)$ by the formula

$$\nu_q(z,\lambda)\,:\,(f_q^{l_1}v_1^q\otimes\ldots\otimes f_q^{l_n}v_n^q)^*\mapsto\sin^l(\pi/p)\,W_{\bar{l}}(t,z,\lambda). \tag{13}$$

We often assume $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant 0}\oplus p\mathbb{Z}$, meaning that for any $k\in\mathbb{Z}_{\geqslant 0}, s\in\mathbb{Z}$, $z_a-z_b+\lambda_a+\lambda_b\neq k+s$.

Lemma 13. *(Cf. Lemma 4.17, Theorem 4.19 in* [TV]*.) The map $\nu_q(z,\lambda)$ is a homomorphism of $U_q\widehat{gl(2)}$ modules. Moreover, if $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant 0}\oplus p\mathbb{Z}$ for all $a>b$, $a,b=1,\ldots,n$, then the map $\nu_q(z,\lambda)$ is an isomorphism.* $\square$

As before, for $z,\lambda\in\mathbb{C}^n$ and permutations $\sigma,\sigma'\in\mathbb{S}^n$, define $z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}\in\mathbb{C}^n$ by equations $(z_\sigma^{\sigma'})_a+(\lambda_\sigma^{\sigma'})_a=z_{\sigma(a)}+\lambda_{\sigma(a)}$ and $(z_\sigma^{\sigma'})_a-(\lambda_\sigma^{\sigma'})_a=z_{\sigma'(a)}-\lambda_{\sigma'(a)}$, $a=1,\ldots,n$.

Lemma 14. *For any permutations $\sigma,\sigma'\in\mathbb{S}^n$, the identity map $\mathrm{Id}:\mathfrak{F}^{\,q}(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})\to\mathfrak{F}^{\,q}(z,\lambda)$ is an isomorphism of $U_q\widehat{gl(2)}$ modules.*

Proof: (Cf. Lemma 2.) The definition of the space $\mathfrak{F}^{\,q}(z,\lambda)$ and of the $U_q\widehat{gl(2)}$ action depends on z,λ only through quantities $\prod_{a=1}^{n}\sin(\pi(u-z_a+\lambda_a)/p)$ and $\prod_{a=1}^{n}\sin(\pi(u-z_a-\lambda_a)/p)$, see (12). $\square$

In what follows we assume that q is not a root of unity, that is $p\notin\mathbb{Q}$.

Lemma 15. *Let $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant 0}\oplus p\mathbb{Z}$, $a,b=1,\ldots,n$. Assume $p\notin\mathbb{Q}$. Then for any permutations $\sigma,\sigma'\in\mathbb{S}^n$, the $U_q\widehat{gl(2)}$ modules $V_q(z,\lambda)$ and $V_q(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$ are isomorphic.*

Proof: The proof of the Lemma is analogues to the proof of Lemma 3. $\square$

Let z,λ,p be as in Lemma 15. Let $(\sigma'\otimes\sigma)(z,\lambda)\,:\,V_q(z,\lambda)\to V_q(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$ be the isomorphism of $U_q\widehat{gl(2)}$ modules such that

$$(\sigma'\otimes\sigma)(z,\lambda):v_1^q\otimes\ldots\otimes v_n^q\mapsto(v_1^q)'\otimes\ldots\otimes(v_n^q)',$$

where $v_i^q,(v_i^q)\prime$ are highest vectors generating $V_q(z_i,\lambda_i), V_q((z_\sigma^{\sigma'})_i,(\lambda_\sigma^{\sigma'})_i)$.

We have

$$(\sigma'\otimes\sigma)(z,\lambda)=(Sh_q(\lambda_\sigma^{\sigma'}))^{-1}\circ(\nu_q(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}))^{-1}\circ\nu_q(z,\lambda)\circ Sh_q(\lambda), \tag{14}$$

where the map ν_q is given by (13).

Example 16. (Cf. Example 4.) Let $n=2$. Consider the transpositions $\sigma=\sigma'=(1,2)$. Then $z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}$ are given by (6).

The isomorpism $\widehat{R}^q(z,\lambda):V_q(z,\lambda)\to V_q(z_{(1,2)}^{(1,2)},\lambda_{(1,2)}^{(1,2)})$, given by formula (14), will be called *the trigonometric* $\widehat{R}^q$ *matrix.*

Example 17. (Cf. Example 5.) Let $n=2$. Consider the case $\sigma=id$, $\sigma'=(1,2)$. Then $z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}$ are given by (7).

We call the isomorphism $N^q(z,\lambda):V_q(z,\lambda)\to V_q(z_{id}^{(1,2)},\lambda_{id}^{(1,2)})$, given by formula (14), *the trigonometric* N^q *matrix.*

Example 18. (Cf. Example 6.) Let $n=2$. Consider the case $\sigma'=id$, $\sigma=(1,2)$. Then $z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}$ are given by (8).

We call the isomorphism $D^q(z,\lambda):V_q(z,\lambda)\to V_q(z_{(1,2)}^{id},\lambda_{(1,2)}^{id})$, given by formula (14), *the trigonometric* D^q *matrix.*

We describe properties of the trigonometric $\widehat{R}^q$, D^q and N^q matrices in the next Section.

3.6. **Properties of R^q, N^q and D^q matrices.** Let n be a natural number. Let $q=\exp(\pi i/p)$ be a complex number, not a root of unity. Let $z,\lambda\in\mathbb{C}^n$ be such that $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant 0}\oplus p\mathbb{Z}$, $a,b=1,\dots,n$.

Lemma 19. *The group $\mathbb{S}^n\times\mathbb{S}^n$ transitively acts on the family of isomorphic $U_q\widehat{gl(2)}$ modules $\{V_q(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}),\sigma,\sigma'\in\mathbb{S}^n\}$, $(\tau'\times\tau)(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}):V_q(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})\to V_q(z_{\tau\sigma}^{\tau'\sigma'},\lambda_{\tau\sigma}^{\tau'\sigma'})$. The action of generators is given by*

$$(a,a+1)\times id=N^q_{a,a+1},\qquad id\times(a,a+1)=D^q_{a,a+1},$$

where $N^q_{a,a+1}$, $D^q_{a,a+1}$ are the trigonometric N^q and D^q matrices acting on the ath and $(a+1)$st factors. We also have

$$N^q_{a,a+1}(z^{id}_{(a,a+1)},\lambda^{id}_{(a,a+1)})D^q_{a,a+1}(z,\lambda)=D^q_{a,a+1}(z^{(a,a+1)}_{id},\lambda^{(a,a+1)}_{id})N^q_{a,a+1}(z,\lambda)=\widehat{R}^q_{a,a+1}(z,\lambda).$$

Proof: The Lemma follows from the definitions of $\widehat{R}^q$, D^q and N^q matrices. $\square$

Let $n=2$. Consider the $U_qsl(2)$ Verma modules $V_q(\lambda_1),V_q(\lambda_2)$ of highest weights $q^{\lambda_1},q^{\lambda_2}$ with highest vectors v_1^q,v_2^q. We have a decomposition of $U_qsl(2)$ modules

$$V_q(\lambda_1)\otimes V_q(\lambda_2)=\bigoplus_{l=0}^{\infty}V_q(\lambda_1+\lambda_2-l).$$

We choose singular vectors $v^q_{(\lambda_1,\lambda_2;\,l)}\in V_q(\lambda_1)\otimes V_q(\lambda_2)$ generating $V_q(\lambda_1+\lambda_2-l)$, so that in the standard basis the coefficient of $f_q^lv_1^q\otimes v_2^q$ is $(B_l^q(\lambda_1))^{-1}$:

$$e_qv^q_{(\lambda_1,\lambda_2;\,l)}=0,\qquad v^q_{(\lambda_1,\lambda_2;\,l)}=(B_l^q(\lambda_1))^{-1}\,f_q^lv_1^q\otimes v_2^q+\sum_{a=1}^{l}d_a^l(\lambda)f_q^{l-a}v_1^q\otimes f_q^av_2^q,$$

$l=0,1,\dots$, where $B_l^q(\lambda)=B_\lambda^q(f_q^lv^q,f_q^lv^q)$ is the value of the quantum Shapovalov form, see (9), and $d_a^l(\lambda)$ are complex numbers.

Theorem 20. *(Spectral decomposition.) Let $A^q : V_q(z,\lambda) \to V_q(u,\omega)$ be either $\widehat{R}^q$, D^q or N^q matrix, where $u, \omega \in \mathbb{C}^2$ are the parameters of the target module. Then A^q commutes with the $U_q sl(2)$ action and maps*

$$A^q \ : \ v^q_{(\lambda_1,\lambda_2;\,l)} \ \mapsto \ \frac{B^q_l((\omega^{id}_{(1,2)})_1)}{B^q_l((\lambda^{id}_{(1,2)})_1)}\, v^q_{(\omega_1,\omega_2;\,l)},$$

where $B^q_l(\lambda) = B^q_\lambda(f^l v, f^l v)$ is the value of the quantum Shapovalov form, see (9).

Proof: In the case of $\widehat{R}^q$ matrix the spectral decomposition is well known, see Proposition 12.5.6 in [CP] and formula (3.16) in [TV]. In the two other cases the proof is similar. $\square$

Note the difference in Theorems 8 and 20 due to the opposite choice of the comultiplications, see (2) and (11).

Let n be any natural number, $z, \lambda \in \mathbb{C}^n$. Let $q = \exp(\pi i/p) \in \mathbb{C}$ be not a root of unity. We say that the parameters $z, \lambda \in \mathbb{C}^n$ are *in a trigonometric resonance* if the $U_q\widehat{gl(2)}$ module $U_q(z,\lambda)$ is reducible. We say that the parameters $z, \lambda \in \mathbb{C}^n$ are *in the first trigonometric resonance* if there exists a unique pair of indices $a, b \in \{1, \ldots, n\}$ such that $z_a - z_b + \lambda_a + \lambda_b \in \mathbb{Z}_{\geqslant 0} \oplus p\mathbb{Z}$ and $a \geqslant b$. We say that the parameters z, λ are *in the second trigonometric resonance* if there exists a unique pair of indices $a, b \in \{1, \ldots, n\}$ such that $z_a - z_b + \lambda_a + \lambda_b \in \mathbb{Z}_{\geqslant 0} \oplus p\mathbb{Z}$ and $a < b$.

Let z, λ be in the first trigonometric resonance. Then the $U_q\widehat{gl(2)}$ module $V_q(z,\lambda)$ has a unique nontrivial submodule. The submodule is isomorphic to a tensor product $V_q(z',\lambda')$ of quantum evaluation Verma modules with parameters $z', \lambda' \in \mathbb{C}^n$, $(z',\lambda') \neq (z,\lambda)$. For example, if $2\lambda_1 = k \in \mathbb{Z}_{\geqslant 0}$, i.e. $a = b = 1$, then $z' = z$, $\lambda' = (-\lambda_1 - 1, \lambda_2, \ldots, \lambda_n)$.

Let z, λ be in the second trigonometric resonance. Then the $U_q\widehat{gl(2)}$ module $V_q(z,\lambda)$ has a unique nontrivial submodule. The factor module is isomorphic to a tensor product $V_q(z',\lambda')$ of quantum evaluation Verma modules with parameters $z', \lambda' \in \mathbb{C}^n$, $(z',\lambda') \neq (z,\lambda)$. For example, if $z_1 - z_2 + \lambda_1 + \lambda_2 = k \in \mathbb{Z}_{\geqslant 0}$, i.e. $a = 1$, $b = 2$, then $z' = z^{id}_{(12)}$, $\lambda' = (-(\lambda^{id}_{(12)})_1 - 1, (\lambda^{id}_{(12)})_2, \ldots, (\lambda^{id}_{(12)})_n)$, see Section 12.2 in [CP] and references therein.

Lemma 21. *Let $z, \lambda \in \mathbb{C}^n$ and $z^{\sigma'}_\sigma, \lambda^{\sigma'}_\sigma \in \mathbb{C}^n$ be either both in the first trigonometric resonance or both in the second trigonometric resonance. Then the map $(\sigma' \times \sigma)(z,\lambda) : V_q(z,\lambda) \to V_q(z^{\sigma'}_\sigma, \lambda^{\sigma'}_\sigma)$ is a well defined isomorphism of $U_q\widehat{gl(2)}$ modules. In particular, it maps the submodule $V_q(z',\lambda') \subset V_q(z,\lambda)$ onto the submodule $V_q((z^{\sigma'}_\sigma)', (\lambda^{\sigma'}_\sigma)') \subset V_q(z^{\sigma'}_\sigma, \lambda^{\sigma'}_\sigma)$. Moreover, the map $(\sigma' \times \sigma)(z,\lambda)$ restricted to the submodule $V_q(z',\lambda')$ coincides with the map $(\sigma' \times \sigma)(z',\lambda')$ up to a non-zero scalar multiplier depending on the choice of the inclusions $V_q(z',\lambda') \hookrightarrow V_q(z,\lambda)$ and $V_q((z^{\sigma'}_\sigma)', (\lambda^{\sigma'}_\sigma)') \hookrightarrow V_q(z^{\sigma'}_\sigma, \lambda^{\sigma'}_\sigma)$.*

Proof: Lemma 21 follows from Theorem 20. $\square$

Lemma 22. *Let $z, \lambda \in \mathbb{C}^n$ be in the first trigonometric resonance and $z^{\sigma'}_\sigma, \lambda^{\sigma'}_\sigma \in \mathbb{C}^n$ be in the second trigonometric resonance. Then the map $(\sigma' \times \sigma)(z,\lambda) :\ V(z,\lambda) \to V(z^{\sigma'}_\sigma, \lambda^{\sigma'}_\sigma)$ is a well defined homomorphism of $U_q\widehat{gl(2)}$ modules. The kernel of this homomorphism is the submodule $V_q(z',\lambda') \subset V(z,\lambda)$ and the image is the proper submodule in $V_q(z^{\sigma'}_\sigma, \lambda^{\sigma'}_\sigma)$.*

Proof: Lemma 22 follows from Theorem 20. $\square$

Lemma 23. *Let $z,\lambda\in\mathbb{C}^n$ be in the second trigonometric resonance and $z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'}\in\mathbb{C}^n$ be in the first trigonometric resonance. Then the map $(\sigma'\times\sigma)(\tilde z,\tilde\lambda):\ V_q(\tilde z,\tilde\lambda)\to V_q(\tilde z_\sigma^{\sigma'},\tilde\lambda_\sigma^{\sigma'})$ has a simple pole at $\tilde z=z,\tilde\lambda=\lambda$. The residue $Res:=\mathrm{res}_{\tilde z=z,\,\tilde\lambda=\lambda}(\sigma'\times\sigma)(\tilde z,\tilde\lambda)$ is a well defined homomorphism of $U_q\widehat{gl(2)}$ modules. The kernel of this homomorphism is the nontrivial submodule U_q of $V_q(z,\lambda)$ and the image is the submodule $V_q((z_\sigma^{\sigma'})',(\lambda_\sigma^{\sigma'})')\subset V_q(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$. Thus, up to a scalar multiplier, depending on the choice of the factor map $V_q(z,\lambda)\to V_q(z',\lambda')\simeq V_q(z,\lambda)/U_q$, and the inclusion $V_q((z_\sigma^{\sigma'})',(\lambda_\sigma^{\sigma'})')\hookrightarrow V_q(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$, the map Res defines a homomorphism $V_q(z,\lambda)/U_q\simeq V_q(z',\lambda')\to V_q((z_\sigma^{\sigma'})',(\lambda_\sigma^{\sigma'})')$. The scalar multiplier can be chosen so that the map Res coincides with the isomorphism $(\sigma'\times\sigma)(z',\lambda')$.*

Proof: Lemma 23 follows from Theorem 20. □

3.7. **The first resonance in the trigonometric hypergeometric space.** Let $z,\lambda\in\mathbb{C}^n$. Let $q=\exp(\pi i/p)$ be a complex number, not a root of unity. Suppose z,λ are in the first trigonometric resonance. We have $z_a-z_b+\lambda_a+\lambda_b=k$ for some $k\in\mathbb{Z}_{\geqslant 0}$, $a,b\in\{1,\dots,n\}$, $a\geqslant b$. Choose $\sigma,\sigma'\in\mathbb{S}^n$, such that $\sigma(a)=\sigma'(b)=1$. Then, by Lemma 21, the map $\sigma'\times\sigma$ is a well defined isomorphism and $2(\lambda_\sigma^{\sigma'})_1=k$.

Let $z,\lambda\in\mathbb{C}^n$ and $2\lambda_1=k$, $k\in\mathbb{Z}_{\geqslant 0}$. As before, set $z'=z$, $\lambda'=(-\lambda_1-1,\lambda_2,\dots,\lambda_n)$.

Define a linear map $\iota_q^*(z,\lambda):\mathfrak{F}^q(z,\lambda)\to\mathfrak{F}^q(z',\lambda')$ as follows. For a function $f_q(t_1,\dots,t_l)\in\mathfrak{F}^q_l(z,\lambda)$, set

$$\tilde f_q(t_1,\dots,t_l)=f_q(t_1,\dots,t_l)\,\sin(\pi(t_1-z_1-\lambda_1)/p)\prod_{a=1}^{k+1}\prod_{b=k+2}^{l}\frac{\sin(\pi(t_a-t_b+1)/p)}{\sin(\pi(t_a-t_b-1)/p)},\ l>k,$$

and $\tilde f=0$ if $l\leqslant k$. Define

$$(\iota_q^*f_q)(t_1\dots,t_{l-k-1})=\tilde f_q(z_1+\lambda_1,z_1+\lambda_1-1,\dots,z_1-\lambda_1,t_1,\dots,t_{l-k-1}).$$

Theorem 24. *The map $\iota_q^*(z,\lambda):\mathfrak{F}^q(z,\lambda)\to\mathfrak{F}^q(z',\lambda')$ is a surjective homomorphism of $U_q\widehat{gl(2)}$ modules.*

Proof: (Cf. Theorem 12.) The map ι_q^* is well defined, it is a surjection, see the definition of the trigonometric hypergeometric space.

If $t_a=z_1+\lambda_1-a+1$, $a=1,\dots,k+1$, then we have the following identities:

$$\prod_{a=1}^{k+1}\frac{\sin(\pi(u-t_a+1)/p)}{\sin(\pi(u-t_a)/p)}\prod_{a=1}^{n}\sin(\pi(u-z_a-\lambda_a)/p)\ =\ \prod_{a=1}^{n}\sin(\pi(u-z_a'-\lambda_a')/p),$$

$$\prod_{a=1}^{k+1}\frac{\sin(\pi(t_a-u+1)/p)}{\sin(\pi(t_a-u-1)/p)}\prod_{a=1}^{n}\frac{\sin(\pi(u-z_a+\lambda_a)/p)}{\sin(\pi(t-z_a-\lambda_a)/p)}\ =\ \prod_{a=1}^{n}\frac{\sin(\pi(u-z_a'+\lambda_a')/p)}{\sin(\pi(u-z_a'-\lambda_a')/p)}.$$

It is a straightforward calculation to check that the map ι_q^* commutes with the $U_q\widehat{gl(2)}$ action given by (12), using the above identities. □

Let $z,\lambda\in\mathbb{C}^n$ and $2\lambda_1=k\in\mathbb{Z}_{\geqslant 0}$. Assume $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant 0}\oplus p\mathbb{Z}$ for all $a>b$, $a,b=1,\dots,n$. Then by Lemma 13,

$$\mathfrak{F}^q(z,\lambda)\simeq V_q^*(z,\lambda)\simeq V_q^*(z_1,\lambda_1))\otimes V_q^*((z_2,\dots,z_n),(\lambda_2,\dots,\lambda_n)),$$

$$\mathfrak{F}^{\,q}(z',\lambda') \simeq V_q^*(z',\lambda') \simeq V_q^*(z_1',\lambda_1') \otimes V_q^*((z_2,\dots,z_n),(\lambda_2,\dots,\lambda_n)).$$

The $U_q\widehat{gl(2)}$ module $V_q^*(z_1,\lambda_1)$ has a submodule, isomorphic to $(k+1)$-dimensional irreducible evaluation module $L_q(z_1,\lambda_1)$. Then $\iota_q^*(z,\lambda) = (\iota_q^*)_1 \otimes \mathrm{Id}$, where

$$(\iota_q^*)_1 : V_q^*(z_1,\lambda_1) \to V_q^*(z_1,\lambda_1)/L_q(z_1,\lambda_1) \simeq V_q^*(z_1',\lambda_1')$$

is the factorization morphism, mapping $f_q^{k+1} v_1^q \otimes v_2^q \otimes \ldots \otimes v_n^q$ to $D_q(k)(v_1^q)' \otimes \ldots (v_n^q)'$. The constant $D_q(k)$ is given by

$$D_q(k) = \sin^{k+1}(\pi/p) \prod_{a=2}^{k+1} \frac{1}{\sin(\pi(t_a - z_1 - \lambda_1)/p)} \prod_{1\leqslant a<b\leqslant k+1} \frac{\sin(\pi(t_a-t_b)/p)}{\sin(\pi(t_a-t_b+1)/p)},$$

where we set $t_a = z_1 + \lambda_1 - a + 1$, $a = 1,\dots,k+1$.

4. Hypergeometric pairing and its properties

4.1. **The qKZ equation.** Fix complex numbers $q = \exp(\pi i/p), \kappa = \exp(\mu)$, where we assume $\mathrm{Re}\, p < 0$, $0 < \mathrm{Im}\, \mu < 2\pi$. Let $\lambda \in \mathbb{C}^n$. The rational $\widehat{R}$ matrix $\widehat{R}((z_i,z_j),(\lambda_i,\lambda_j))$, depends on z_i, z_j only through the difference $z_i - z_j$, see Theorem 8. The operator $R(z_i - z_j) = P\widehat{R}((z_i,z_j),(\lambda_i,\lambda_j)) \in \mathrm{End}(V(\lambda_i)\otimes V(\lambda_j))$, where P is the operator of permutation of the factors, will be called the rational R matrix.

The rational quantized Knizhnik-Zamolodchikov equation (qKZ) with values in a tensor product of $sl(2)$ Verma modules $V(\lambda_1)\otimes\ldots\otimes V(\lambda_n)$ is a system of difference equations for a function $\Psi(z_1,...,z_n) \in V(\lambda_1)\otimes\ldots\otimes V(\lambda_n)$. The system of equations has the form

$$\Psi(z_1,\dots,z_m+p,\dots,z_n) = H_m(z,\lambda)\;\Psi(z_1,\dots,z_n), \qquad m = 1,\dots,n,$$

$$\begin{aligned} H_m(z,\lambda) \;&=\; R_{m,m-1}(z_m - z_{m-1} + p)\ldots R_{m,1}(z_m - z_1 + p)\,\times \\ &\times\;\; \kappa^{-h_m} R_{m,n}(z_m - z_n)\ldots R_{m,m+1}(z_m - z_{m+1}), \end{aligned} \tag{15}$$

where h_m is the operator $h \in sl(2)$ acting in the m-th factor, $R_{i,j}(z_i - z_j)$ is the rational R matrix acting in the i-th and j-th factors of the tensor product.

4.2. **The hypergheometric pairing.** Let $z = (z_1,\dots,z_n) \in \mathbb{C}^n$, $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n$, $t = (t_1,\dots,t_l) \in \mathbb{C}^l$. Let $l \in \mathbb{Z}_{\geqslant 0}$. The *phase function* is defined by the following formula:

$$\Phi_l(t,z,\lambda) = \frac{1}{l!}\,\exp(\mu\sum_{a=1}^{l} t_a/p) \prod_{a=1}^{n}\prod_{b=1}^{l} \frac{\Gamma((t_b - z_a + \lambda_a)/p)}{\Gamma((t_b - z_a - \lambda_a)/p)} \prod_{1\leqslant a<b\leqslant l} \frac{\Gamma((t_a - t_b - 1)/p)}{\Gamma((t_a - t_b + 1)/p)}.$$

Assume that the parameters $z,\lambda \in \mathbb{C}^n$ satisfy the condition $\mathrm{Re}\,(z_i + \lambda_i) < 0$ and $\mathrm{Re}\,(z_i - \lambda_i) > 0$ for all $i = 1,\dots,n$. For functions $w(t,z,\lambda) \in \mathfrak{F}_l(z,\lambda)$, and $W(t,z,\lambda) \in \mathfrak{F}_l^{\,q}(z,\lambda)$, define the *hypergeometric integral* $I(w,W)(z,\lambda)$ by the formula

$$I(w,W)(z,\lambda) = \int\limits_{\substack{\mathrm{Re}\, t_i = 0,\\ i=1,\dots,l}} \Phi_l(t,z,\lambda) w(t,z,\lambda) W(t,z,\lambda)\, d^l t, \tag{16}$$

where $d^l t = dt_1 \ldots dt_l$.

The hypergeometric integral for generic z, λ is defined by analytic continuation with respect to z, λ and has the form

$$I(w, W)(z, \lambda) = \int\limits_{\mathcal{K}(z,\lambda;p)} \Phi_l(t, z, \lambda) w(t, z, \lambda) W(t, z, \lambda)\, d^l t,$$

for some suitable contour of integration $\mathcal{K}(z, \lambda; p) \subset \mathbb{C}^l$, see [MV1]. The hypergeometric integral $I(w, W)(z, \lambda)$ is a univalued meromorphic function of variables z, λ.

Define *the hypergeometric pairing* $\mathcal{H}(z, \lambda) : \mathfrak{F}^q(z, \lambda) \otimes \mathfrak{F}(z, \lambda) \to \mathbb{C}$ by

$$\mathcal{H}(z, \lambda) \,:\, W(t, z, \lambda) \otimes w(t, z, \lambda) \mapsto I(w, W)(z, \lambda),$$

if $w(t, z, \lambda) \in \mathfrak{F}_l(z, \lambda)$, $W(t, z, \lambda) \in \mathfrak{F}^q_l(z, \lambda)$, and let $\mathcal{H}(W \otimes w) = 0$ if $w \in \mathfrak{F}_l(z, \lambda)$, $W \in \mathfrak{F}^q_k(z, \lambda)$, $k \neq l$.

Assume $p \notin \mathbb{Q}$ and $z_a - z_b + \lambda_a + \lambda_b \notin \mathbb{Z}_{\geqslant 0} + p\mathbb{Z}$, $a, b = 1, \dots n$. Then by Lemmas 1, 13, we have a map

$$\mathcal{H}(z, \lambda) \,:\, V^*_q(z, \lambda) \otimes V^*(z, \lambda) \to \mathbb{C}.$$

Define *the qKZ map*,

$$\begin{aligned} \mathrm{qKZ}(z, \lambda) \;&:\; V_q(z, \lambda) \to V(z, \lambda), \\ \mathrm{qKZ}(z, \lambda) \;&:\; f_q^{l_1} v_1^q \otimes \ldots \otimes f_q^{l_n} v_n^q \mapsto \Psi_{\bar{l}}(z, \lambda), \\ \Psi_{\bar{l}}(z, \lambda) \;&=\; \sum_{\bar{m} \in \mathcal{Z}^l_n} B^q_{\bar{l}}(\lambda)\, I(w_{\bar{m}}, W_{\bar{l}})(z, \lambda)\; f^{m_1} v_1 \otimes \ldots \otimes f^{m_n} v_n, \end{aligned} \tag{17}$$

where $B^q_{\bar{l}}(\lambda) = B^q(f_q^{l_1} v_1^q \otimes \ldots \otimes f_q^{l_n} v_n^q, f_q^{l_1} v_1^q \otimes \ldots \otimes f_q^{l_n} v_n^q)$ is the value of the quantum Shapovalov form, see (9).

We have $\mathrm{qKZ}(z, \lambda) = \tilde{\mathcal{H}}(z, \lambda) \circ Sh_q(\lambda)$, where $\tilde{\mathcal{H}}(z, \lambda) : V^*_q(z, \lambda) \to V(z, \lambda)$ is the map dual to the hypergeometric pairing.

The meromorphic functions $\{\Psi_{\bar{l}}(z, \lambda),\ \bar{l} \in \mathbb{Z}^n_{\geqslant 0}\}$ form a basis of solutions of the rational qKZ equation, see Corollary 5.25 in [TV]. In particular, if $p \notin \mathbb{Q}$ and $z_a - z_b + \lambda_a + \lambda_b \notin \mathbb{Z}_{\geqslant 0} + p\mathbb{Z}$, $a, b = 1, \dots n$, then the qKZ map is an isomorphism of vector spaces, meromorphic with respect to variables z, λ.

4.3. **The qKZ map and the $\widehat{R}$, D, N matrices.**

Theorem 25. *Let $\Psi_{\bar{l}}(z, \lambda)$ be a hypergeometric solution of the qKZ equation with values in $V(z, \lambda)$ given by (17). Then for any permutations $\sigma, \sigma' \in \mathbb{S}^n$, the function $(\sigma' \times \sigma)\,(\Psi_{\bar{l}}(z, \lambda))$ is a solution of the qKZ equation with values in $V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$. Moreover, the following diagram is commutative:*

$$\begin{array}{ccc} V_q(z, \lambda) & \xrightarrow{\mathrm{qKZ}(z,\lambda)} & V(z, \lambda) \\ \downarrow\ \sigma' \times \sigma & & \downarrow\ \sigma' \times \sigma \\ V_q(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma}) & \xrightarrow{\mathrm{qKZ}(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})} & V(z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma}). \end{array} \tag{18}$$

Proof: We have $\Phi_l(t, z, \lambda) = \Phi_l(t, z^{\sigma'}_{\sigma}, \lambda^{\sigma'}_{\sigma})$. Theorem 25 follows from the definitions of all participating maps and Lemmas 2 and 14. $\square$

4.4. **The extended qKZ equation.** Consider the space $\mathbb{C}^{2n}$ with coordinates (z,λ), $z,\lambda\in\mathbb{C}^n$. For permutations $\sigma',\sigma\in\mathbb{S}^n$, introduce shifts $T_\sigma^{\sigma'}(p):\mathbb{C}^{2n}\to\mathbb{C}^{2n}$ by the formula

$$T_\sigma^{\sigma'}(p)(z,\lambda))=((\sigma')^{-1}\times(\sigma)^{-1})\circ T(p)\circ(\sigma'\times\sigma)\,(z,\lambda),\qquad z,\lambda\in\mathbb{C}^n,$$

where $(\sigma'\times\sigma)\,(z,\lambda)=(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'})$, and $T(p)(z,\lambda)=((z_1+p,z_2,\dots,z_l),(\lambda_1,\dots,\lambda_n))$.

Introduce a system of difference equations for a function $\Psi(z,\lambda)\in V(z,\lambda)$ by

$$\Psi(T_\sigma^{\sigma'}(p)(z,\lambda))=H_\sigma^{\sigma'}(z,\lambda;p,\mu)\Psi(z,\lambda),\tag{19}$$

where

$$H_\sigma^{\sigma'}(z,\lambda;p,\mu)=((\sigma')^{-1}\times(\sigma)^{-1})\circ H_1(z_\sigma^{\sigma'},\lambda_\sigma^{\sigma'};p,\mu)\circ(\sigma'\times\sigma)\in\mathrm{End}(V(z,\lambda)),$$

and $H_1(z,\lambda;p,\mu)$ is the first qKZ operator, see (15). We call the system 19 *the extended qKZ equation.*

Example 26. The system (19) contains the qKZ equation (15). Namely, for permutations σ,σ' such that $\sigma=\sigma'$, $\sigma(1)=i$ the equation (19) takes the form $\Psi(z_1,\dots,z_i+p,\dots,z_n,\lambda)=H_i(z,\lambda)\Psi(z,\lambda)$, where $H_i(z,\lambda)$ is given by (15).

Example 27. The shifts $T_\sigma^{\sigma'}(p):\mathbb{C}^{2n}\to\mathbb{C}^{2n}$ preserve the quantity $\lambda_1+\dots+\lambda_n$ and generate a group acting in the space $\mathbb{C}^{2n}$ of parameters (z,λ) isomorpfic to $\mathbb{Z}^{2n-1}$. For example, we have

$$T_{(1,2)}^{id}(z,\lambda)=((z_1+p/2,\ z_2+p/2,\ z_3,\dots,z_n),(\lambda_1-p/2,\ \lambda_2+p/2,\ \lambda_3,\dots,\lambda_n)).$$

Corollary 28. *The extended qKZ equation is a compatible (holonomic) system of difference equations, i.e. for any permutations $\sigma,\sigma',\tau,\tau'\in\mathbb{S}^n$,*

$$H_\sigma^{\sigma'}(T_\tau^{\tau'}(p)(z,\lambda))H_\tau^{\tau'}(z,\lambda)=H_\tau^{\tau'}(T_\sigma^{\sigma'}(p)(z,\lambda))H_\sigma^{\sigma'}(z,\lambda).$$

Moreover, the meromorphic functions $\{\Psi_{\bar l}(z,\lambda),\ \bar l\in\mathbb{Z}_{\geqslant0}^n\}$, where the function $\Psi_{\bar l}(z,\lambda)$ is given by (17), form a basis of solutions of the extended qKZ equation.

Proof: Corollary 28 follows from Theorem 25. $\square$

It would be interesting to find an algebraic proof of Corollary 28.

4.5. **The qKZ map, factormodules and submodules.** Introduce a function

$$C(z,\lambda)=\frac{[k]_q!\,[k+1]_q!}{p^{k+1}\,(k+1)!}\,\Gamma(k/p)\ \exp(\mu\sum_{a=1}^{k+1}t_a/p)\times\tag{20}$$

$$\times\prod_{1\leqslant a<b-1\leqslant k}\frac{\Gamma((t_a-t_b-1)/p)}{\Gamma((t_a-t_b+1)/p)}\prod_{a=2}^{k+1}\frac{\Gamma((t_a-z_1+\lambda_1)/p)}{\Gamma((t_a-z_1-\lambda_1)/p)}\prod_{a=2}^{n}\prod_{b=1}^{k+1}\frac{\Gamma((t_b-z_a+\lambda_a)/p)}{\Gamma((t_b-z_a-\lambda_a)/p)},$$

where we set $t_a=z_1+\lambda_1-a+1$, $a=1,\dots,k+1$, cf. formula (16) in [MV1].

Theorem 29. *Let $z,\lambda\in\mathbb{C}^n$ and $2\lambda_1=k\in\mathbb{Z}_{\geqslant0}$. Assume $z_a-z_b+\lambda_a+\lambda_b\notin\mathbb{Z}_{\geqslant0}\oplus p\mathbb{Z}$ for all $a>b$, $a,b=1,\dots,n$. Let $q=\exp(\pi i/p)$ and $p\notin\mathbb{Q}$. Let $\lambda_1'=-\lambda_1-1$, and $\tilde z=(z_2,\dots,z_n)$, $\tilde\lambda=(\lambda_2,\dots,\lambda_n)$. Then there exist a map $\alpha(z,\lambda):L_q(z_1,\lambda_1)\otimes$*

$V_q(\tilde z,\tilde\lambda) \to L(z_1,\lambda_1)\otimes V(\tilde z,\tilde\lambda)$ such that the following diagram is commutative and its columns and rows form exact sequences:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & L_q(z_1,\lambda_1)\otimes V_q(\tilde z,\tilde\lambda) & \xrightarrow{\alpha(z,\lambda)} & L(z_1,\lambda_1)\otimes V(\tilde z,\tilde\lambda) & \longrightarrow & 0\\
 & & \uparrow Sh_q\otimes \mathrm{Id} & & \uparrow Sh\otimes\mathrm{Id} & & \\
0 & \longrightarrow & V_q(z_1,\lambda_1)\otimes V_q(\tilde z,\tilde\lambda) & \xrightarrow{\mathrm{qKZ}(z,\lambda)} & V(z_1,\lambda_1)\otimes V(\tilde z,\tilde\lambda) & \longrightarrow & 0\\
 & & \uparrow\ \iota_q(z,\lambda)\otimes\mathrm{Id} & & \uparrow\ \iota(z,\lambda)\otimes\mathrm{Id} & & \\
0 & \longrightarrow & V_q(z_1,\lambda_1')\otimes V_q(\tilde z,\tilde\lambda) & \xrightarrow{C(z,\lambda)\,\mathrm{qKZ}(z,\lambda')} & V(z_1,\lambda_1')\otimes V(\tilde z,\tilde\lambda) & \longrightarrow & 0,\\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{21}$$

where $C(z,\lambda)$ is given by (20).

Proof: Consider the hypergeometric integral

$$I(w,W_{\bar l})(z,\lambda) = B^q_{\bar l}(\lambda)\int\limits_{\mathcal K(z,\lambda;p)}\Phi_l(t,z,\lambda)w(t,z,\lambda)W_{\bar l}(t,z,\lambda)\,d^lt, \qquad l_1>k,$$

at $\lambda_1=k/2$. The function $B^q_{\bar l}(\lambda)$ as a function of λ_1 has a zero of the first order. The integral $\int\ \Phi_l(t,z,\lambda)w(t,z,\lambda)W_{\bar l}(t,z,\lambda)\,d^lt$ has a pole of the first order and the residue is

$$\mathrm{res}_{2\lambda_1=k}\int\limits_{\mathcal K(z,\lambda;p)}\Phi_l(t,z,\lambda)w(t,z,\lambda)W_{\bar l}(t,z,\lambda)\,d^lt = \binom{l}{k+1}\times$$

$$\times\int\limits_{\mathcal K(z,\lambda';p)}\mathrm{res}_{t_{k+1}=z_1-\lambda_1}\dots\mathrm{res}_{t_2=z_1+\lambda_1-1}\mathrm{res}_{t_1=z_1+\lambda_1}\Phi_l(t,z,\lambda)w(t,z,\lambda)W_{\bar l}(t,z,\lambda)d^{l-k-1}t,$$

where $\lambda'=(\lambda_1',\lambda_2,\dots,\lambda_n)$, see the proof of Theorem 10 in [MV1]. We have

$$\mathrm{res}_{t_{k+1}=z_1-\lambda_1}\dots\mathrm{res}_{t_2=z_1+\lambda_1-1}\mathrm{res}_{t_1=z_1+\lambda_1}\ \frac{\binom{l}{k+1}\,\Phi_l(t,z,\lambda)}{(t-z_1-\lambda_1)\,\sin(\pi(t-z_1-l_1)/p)} =$$

$$= C(z,\lambda)\,\Phi_{l-k-1}((t_{k+2},\dots,t_l),z,\lambda')\prod_{a=1}^{k+1}\prod_{b=k+2}^{l}\frac{t_a-t_b+1}{t_a-t_b-1}\,\frac{\sin(\pi(t_a-t_b+1)/p)}{\sin(\pi(t_a-t_b-1)/p)}.$$

The commutativity of the bottom square of diagram 21 now follows from Theorems 12 and 24.

There exist the unique map $L_q(z_1,\lambda_1)\otimes V_q(\tilde z,\tilde\lambda)\to L(z_1,\lambda_1)\otimes V(\tilde z,\tilde\lambda)$ such that the digram 21 is commutative. □

Remark. A version of Theorem 29 was implicitely used in [MV1] to construct solutions of the qKZ equation with values in tensor products of finite dimensional representations $L(\lambda_1)\otimes\dots\otimes L(\lambda_n)$, $2\lambda\in\mathbb Z^n_{\geqslant 0}$. □

Corollary 30. *Let $z, \lambda \in \mathbb{C}^n$ be in the first trigonometric resonance and in the first rational resonance. Then the map* $\mathrm{qKZ}(z, \lambda) : V_q(z, \lambda) \to V(z, \lambda)$ *is a well defined isomorphism of linear spaces. Moreover, it maps the submodule* $V_q(z', \lambda') \subset V_q(z, \lambda)$ *onto the submodule* $V(z', \lambda') \subset V(z, \lambda)$. *The map* $\mathrm{qKZ}(z, \lambda)$ *restricted to the submodule* $V_q(z', \lambda')$ *coincides with the map* $\mathrm{qKZ}(z', \lambda')$ *up to a non-zero scalar multiplier depending on the choice of the inclusions* $V_q(z', \lambda') \hookrightarrow V_q(z, \lambda)$ *and* $V(z', \lambda') \hookrightarrow V(z, \lambda)$.

Proof: Corollary 30 follows from Theorem 29 and Lemmas 9, 21. □

Corollary 31. *Let $z, \lambda \in \mathbb{C}^n$ be in the first trigonometric resonance and in the second rational resonance. Then the map* $\mathrm{qKZ}(z, \lambda) : V_q(z, \lambda) \to V(z, \lambda)$ *is a well defined linear map. The kernel of this map is the submodule* $V_q(z', \lambda') \subset V_q(z, \lambda)$ *and the image is the proper submodule in* $V(z, \lambda)$.

Proof: Corollary 31 follows from Theorem 29 and Lemmas 10, 22. □

Corollary 32. *Let $z, \lambda \in \mathbb{C}^n$ be in the second trigonometric resonance and in the first rational resonance. Then the map* $\mathrm{qKZ}(\tilde{z}, \tilde{\lambda}) : V_q(\tilde{z}, \tilde{\lambda}) \to V(\tilde{z}, \tilde{\lambda})$ *has a simple pole at* $\tilde{z} = z, \tilde{\lambda} = \lambda$. *Let Res be the residue of the map* $\mathrm{qKZ}(\tilde{z}, \tilde{\lambda})$ *at* $\tilde{z} = z, \tilde{\lambda} = \lambda$. *The kernel of the map Res is the nontrivial submodule* U_q *of* $V_q(z, \lambda)$ *and the image is the submodule* $V(z', \lambda') \subset V(z, \lambda)$. *Thus, up to a scalar multiplier, depending on the choice of the factor map* $V_q(z, \lambda) \to V_q(z', \lambda') \simeq V_q(z, \lambda)/U_q$, *and the inclusion* $V(z', \lambda') \hookrightarrow V(z, \lambda)$, *the map Res defines a homomorphism* $V_q(z', \lambda') \to V(z', \lambda')$. *The scalar multiplier can be chosen so that the map Res coincides with the isomorphism* $\mathrm{qKZ}(z', \lambda')$.

Proof: Corollary 32 follows from Theorem 29 and Lemmas 11, 23. □

Example 33. Let $z, \lambda \in \mathbb{C}^n$ be in the second trigonometric resonance and in the first rational resonance. Let $(\lambda^{(1,2)}_{id})_1 = z_1 - z_2 + \lambda_1 + \lambda_2 = k \in \mathbb{Z}_{\geqslant 0}$. Denote $u = z^{(1,2)}_{id}, \omega = \lambda^{(1,2)}_{id}$, $\tilde{z} = (z_3, \ldots, z_n)$, $\tilde{\lambda} = (\lambda_3, \ldots, \lambda_n)$. Then there exists the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \uparrow & & \\
 & & L_q(u_1,\omega_1) \otimes V_q(u_2,\omega_2) \otimes V_q(\tilde{z},\tilde{\lambda}) & \xrightarrow{0} & L(u_1,\omega_1) \otimes V(u_2,\omega_2) \otimes V(\tilde{z},\tilde{\lambda}) & & \\
 & & \downarrow & & \uparrow & & \\
 & & V_q(z_1,z_2,\lambda_1,\lambda_2) \otimes V_q(\tilde{z},\tilde{\lambda}) & \xrightarrow{Res} & V(z_1,z_2,\lambda_1,\lambda_2) \otimes V(\tilde{z},\tilde{\lambda}) & & \\
 & & \downarrow & & \uparrow & & \\
0 & \to & V_q(u_1,\omega_1') \otimes V_q(u_2,\omega_2) \otimes V_q(\tilde{z},\tilde{\lambda}) & \xrightarrow{\mathrm{qKZ}} & V(u_1,\omega_1') \otimes V(u_2,\omega_2) \otimes V(\tilde{z},\tilde{\lambda}) & \to & 0. \\
 & & \downarrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Here the left column is an exact short sequence of $U_q\widehat{gl(2)}$ modules and the right column is an exact short sequence of Yangian modules, $Res = \mathrm{res}_{z_1 = z_2 - \lambda_1 - \lambda_2 + k}\, \mathrm{qKZ}(\tilde{z}, \tilde{\lambda})$.

4.6. **Singularities of hypergeometric solutions.**

Theorem 34. *Let p be a complex number such that $\mathrm{Re}\, p < 0$ and $p \notin \mathbb{Q}$. The map $qKZ(z,\lambda) : V_q(z,\lambda) \to V(z,\lambda)$ is a well defined isomorhism of vector spaces for all z, λ except for the following two cases.*

i) The map $qKZ(z,\lambda)$ has a nontrivial simple pole at the hyperplanes $z_i - z_j + \lambda_i + \lambda_j = m - ps$, $m, s \in \mathbb{Z}_{\geqslant 0}$, $i < j$.

ii) The map $qKZ(z,\lambda)$ has a nontrivial kernel at the hyperplanes $z_i - z_j + \lambda_i + \lambda_j = m + ps$, $m, s \in \mathbb{Z}_{\geqslant 0}$, $j < i$.

Proof: By Corollary 5.25 in [TV] the map $\mathrm{qKZ}(z,\lambda)$ is a linear isomorphism if $z_i - z_j + \lambda_i + \lambda_j \notin \mathbb{Z}_{\geqslant 0} + p\mathbb{Z}$.

Consider the case $i = 1$, $j = 2$. By Corollary 32, the map $\mathrm{qKZ}(z,\lambda)$ has a pole at the hyperplane $z_1 - z_2 + \lambda_1 + \lambda_2 = k \in \mathbb{Z}_{\geqslant 0}$. We have

$$(H_1(z,\lambda))^s \mathrm{qKZ}(z,\lambda) = \mathrm{qKZ}(z_1 + sp, z_2, \dots, z_n, \lambda)), \qquad s \in \mathbb{Z},$$

where H_1 is the first qKZ operator given by (15). The operator $H_1(z,\lambda)$ has a nontrivial kernel at the hyperplane $z_1 - z_2 + \lambda_1 + \lambda_2 = k \in \mathbb{Z}_{\geqslant 0}$. It is easy to see that at this hyperplane for generic $z_3, \dots, z_n$, the product $H_1(z,\lambda) qKZ(z,\lambda)$ is a well defined nondegenerate operator.

Note that for generic $z_3, \dots, z_n$, the operator $H_1(z,\lambda)$ is an isomorphism if $z_1 - z_2 + \lambda_1 + \lambda_2 = k + sp$, $s \neq 0, z \in \mathbb{Z}$. Hence, the map $\mathrm{qKZ}(z,\lambda)$ is an isomorpism at $z_1 - z_2 + \lambda_1 + \lambda_2 \in \mathbb{Z} + p\mathbb{Z}_{>0}$ and has a nontrivial pole at $z_1 - z_2 + \lambda_1 + \lambda_2 \in \mathbb{Z} + p\mathbb{Z}_{\leqslant 0}$.

The case of generic $i, j \in \{1, \dots, n\}$ is done similarly. $\square$

Remark. In fact Theorem 34 combined with the results of Section 4.5 allows to describe all the singularities of the hypergeometric solutions. Indeed, consider a hypergeometric solution $\Psi_{\bar{l}}(z,\lambda)$ given by (17). It has poles of the first order at the hyperplanes $z_i - z_j + \lambda_i + \lambda_j = m - ps$, $s \in \mathbb{Z}_{\geqslant 0}$, $m = 0, 1 \dots, l-1$, $i < j$.

Consider the hyperplane $z_1 - z_2 + \lambda_1 + \lambda_2 = m$. The residue $\mathrm{res}_{z_1 = z_2 - \lambda_1 - \lambda_2 + k} \Psi_{\bar{l}}(z,\lambda)$ is a function with values in the proper Yangian submodule $V(u,\omega) \subset V(z,\lambda)$, cf. Example 33. In fact this function is a linear combination of hypergeometric solutions $\Psi_{\bar{m}}(u,\omega)$ with $m = l - k - 1$ integrations. This linear combination is determined by the image of the vector $f^{l_1} v_1 \otimes \dots \otimes f^{l_n} v_n \in V_q(z,\lambda)$ under the factorization map of $U_q\widehat{gl(2)}$ modules $V_q(z,\lambda) \to V_q(u,\omega)$.

The residue of the hypergeometric function $\Psi_{\bar{l}}(z,\lambda)$ at the hyperplanes $z_1 - z_2 + \lambda_1 + \lambda_2 = m - ps$, $s = 1, 2, \dots$, is computed by applying s times the qKZ operator H_2 to the residue of $\Psi_{\bar{l}}(z,\lambda)$ at the hyperplane $z_1 - z_2 + \lambda_1 + \lambda_2 = m$. $\square$

Remark. In this paper we treated the case $|\kappa| \neq 1$. The case $\kappa = 1$ is important for applications and is done in a similar way. In this case the qKZ map is defined on the subspaces of singular vectors,

$$\mathrm{qKZ}(z,\lambda;\, p, \kappa = 1) : (V^q(z,\lambda))^{sing} \to (V(z,\lambda))^{sing},$$

where $(V^q(z,\lambda))^{sing} = \mathrm{Ker}\ e_q \subset V_q(z,\lambda)$, $(V(z,\lambda))^{sing} = \mathrm{Ker}\ e \subset V(z,\lambda)$. We get the same statements as in Corollaries 30-32, where all the tensor products of modules are replaced with the corresponding subspaces of singular vectors.

However, in the case $\kappa = 1$, the map $\mathrm{qKZ}(z,\lambda)$ has nontrivial degeneracies which do not come from singularities of the qKZ equation, see [MV2],[MV3]. $\square$

Remark. One can prove a statement similar to Corollary 32 for the case $2\lambda \in \mathbb{Z}^n_{\geqslant 0}$ with Verma modules $V(\lambda_i), V_q(\lambda_i)$ replaced by finite dimensional modules $L(\lambda_i), L_q(\lambda_i)$. In this case, it is sufficient to assume that q is not a root of unity of a small order. Namely, it is sufficient to assume $q^a \neq 1$ for $a = 1, \dots, \max(2\lambda_1, \dots, 2\lambda_n, k, 2\lambda_1 + 2\lambda_2 - k)$. $\square$

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